THE DYNAMICS OF DECISION PROCESSES

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i. Forward

This book is part of an ongoing project to revolutionize the way we talk about decision processes. Others have started this conversation. In volume 1, my contribution was the proposal that the dynamic properties of decision processes are best described by the mathematics of differential geometry (Thomas G. H., 2006). Illustrative examples using streamlines showed that decision processes would exhibit not just persistent behaviors but oscillatory behaviors analogous to weather patterns with hurricanes, tornadoes and other vortical behaviors.

After completion of that project, it became obvious to me that there in fact might be additional solutions of interest. Just as volume 1 extended the essentially DC behavior of game theory to an AC version of discrete steady state solutions, there might also be interest in further extensions, using the same analogy, to transient behaviors. On further reflection, it appeared that the discrete solutions obtained in volume 1 are analogous to resonant phenomena in physical systems. In addition there should be transient and steady state AC solutions of any frequency that are driven by the initial and boundary conditions of the system. It is the intent of this volume to focus on such solutions and show that they do exist.

To obtain these additional solutions, readers may need additional mathematical tools and background. Therefore some background will be provided here from the areas of physics, differential geometry and engineering. In addition, we supply further examples from the decision process literature to show how these additional behaviors might provide meaningful insight. As a further extension of the first volume, we also expand the vocabulary to cover these new behaviors. In short, we develop an engineering discipline of decision processes.

This current project also grew out of a desire to further the relevance of volume 1 to decision processes and to create a pedagogy that would allow potential students to learn and apply these ideas to realistic situations. This is consistent with providing a sound mathematical basis and in the process, further developing the ideas of deterministic decision processes. Thus I describe a complete, comprehensive and quantitative theory of the time dependence of decision processes. To obtain the new behaviors of interest, I apply this theory with techniques used in electrical engineering, meteorology and general relativity to gain insights into such diverse issues as the prisoner's dilemma in game theory and ethical issues such as the tragedy of the commons.

The goal of the theory is to cover all aspects of economic behaviors. In this volume, I review briefly the static and equilibrium behaviors described by game theory, which are analogous to DC circuit behaviors. I extend such static behaviors to steady-state behaviors analogous to AC circuit behaviors. The full treatment includes additional dynamic behaviors, including transient dynamic behaviors.

As with any scientific theory, it is crucial to compare results against experimental facts. This book provides the necessary tools to carry that out. I show how numerical solutions to the decision process equations can be obtained along with numerous illustrative examples. The equations can be characterized as Einstein-like equations and so the techniques and solutions obtained also may be of interest to those in other fields such as relativity, gauge theory and differential geometry.

I make no assumption that the reader is familiar with the ideas of differential geometry or with their applications in physics. I assume that the audience may not be familiar with the vast literature on game theory, which is a rather specialized aspect of economic theory. This presents challenges that are not unfamiliar to those in engineering and science. On the one hand we must provide motivation to the reader to make the effort to learn the new techniques and on the other hand we must provide adequate tools and training for the reader to be able to apply and use those techniques on interesting problems. The payoff in the end is the possibility of being able to vastly improve the readers understanding of real-world decision processes.

Thus the goal for this book is to select and bring the most relevant components of the vast literature on differential geometry including the physics of gauge theories to the reader interested in decision processes. The approach is similar to bringing the ideas of physics to engineering students who want to apply the ideas to realistic problems. I require that the student have a minimal background in mathematics and science corresponding to the same requirements for a sophomore level engineering student. The goal in this book is to make the ideas accessible to such a student. I do this at three levels. I provide an exposition of the ideas, I provide a set of problems to enhance the student's understanding of the material and I provide numerical examples with the help of Mathematica. The latter provides an important visual tool for making the ideas come alive.

As the purpose of this book is to provide the necessary mathematical and engineering training to both understand and apply the approach, out of necessity it is cross disciplinary and hopefully sheds light on the areas from which it borrows.

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ii. Preface

Decision making is a distinctly human process. We constantly make decisions. What time will we get up in the morning? What activities will we do during the day? What product will we make and sell? What products will we buy? What will we eat for dinner? Who will we vote for? How will we defend against terrorists? The list is endless, covering the entire range of human activities. Our rather surprising ability is that we can make decisions without effort. We are able to sift through the possible choices without explicitly enumerating them. We live with the consequence of our decision and seem to learn from our mistakes. We seem to be able to recognize the key patterns of decision making and take appropriate action. We are firm believers that our decisions are optimal and our decision process adequate for our needs.

Given this state of affairs, is there any need to improve our decision making process? I assert that the answer is yes. There are times when our decision process leads to disaster. In economics, apparently sound business decisions can lead to recessions and depressions. In business dealings, talented organizations behaving rationally still can build products that don't sell, can fall behind on their delivery schedules and can run out of money. As individuals our successes can lead us to over indulge; we can become over-weight or can run up large credit card debts.

It may be that we are adept at responding quickly to certain situations such as predator attack, situations that are local in time and space. We may not be as adept at recognizing situations that involve correlations of events at large distances or long time durations. We make no distinction to what might be distinct decision processes. An engineering analog might be helpful. We might be good at stacking a few building blocks on top of one another, but less adept at building towers of 20 or 200 or 2000 such blocks. We have developed engineering practices, *i.e.* rules of behaviors, which provide guidelines for building large or complex structures. These guidelines help us build structures that would have been impossible to imagine before engineering. I grew up in Los Angeles that limited buildings to 12 stories because of fear of earthquakes but today, the city allows significantly taller skyscrapers that have a better chance of surviving than structures built a hundred years ago.

To address these issues, I rely not on the usual stochastic approach but on a deterministic and physical model of decisions, a model that incorporates both the competitive aspect of decisions as well as the temporal aspects of a causal world. Not everyone may agree with this hypothesis, yet ultimately agreement or disagreement rests not a belief but on whether the hypothesis provides an explanation of known data. Further, a useful theory is ultimately one that gives insight into behaviors that have not yet happened; insight that is both qualitative and quantitative.

It may be helpful to say what the reader may or may not find in this book. The exposition of this theory requires making a set of decisions: what material should be included and what examples should be chosen. I wish I could argue that the theory provides that guidance. In point of fact, I can't make such a bootstrap and my choices are motivated by my own experiences as well as what I have learned from a vast number of people that I have met over time.

Based on my experiences as a research physicist I have chosen what I think are the important mathematical and physical topics. Based on my experiences as an engineering professor I have chosen problems that I believe provide pedagogical support. And based on my experiences from industry I have chosen examples that I think are important and relevant. But by far I owe the greatest debt to those that have shared their experiences with me. This has occurred over many years and I am not able to provide a detailed accounting. But I am able to point to a few people who have been especially helpful for this phase of the project.

I have enjoyed, an am grateful for, an ongoing relationship with the Partnership Program Support at Wolfram Research. Lou Kauffman strongly supported the first phase of this research and provided in this new phase, access and feedback from his students. Gordon Kane provided feedback on the physics and pedagogy based on a draft of the manuscript. Jack Behrend read and provided edits of countless early drafts for which I am grateful. I received useful and supportive feedback on my philosophical stance from

Steve Rosen. Lisa Maroski and Helen Kessler have held me accountable for my project and goals as well as providing feedback on specific sections. They have honed my understanding of Systems Dynamics as a deterministic approach applied to a variety of real world applications. Keelan Kane collaborated with me on some paradoxical problems in the literature on the prisoner's dilemma, which led to chapter 5. Col. Edward Sobiesk introduced me to the new field of Network Science, and the interest at West Point in applications. My Colleagues at MSOE have patiently listened to my attempts to clarify my ideas. I am particularly grateful to Bharathwaj Muthuswamy and Jovan Jevtic and our weekly meeting to discuss a variety of research topics including this one.

G. H. Thomas

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1 Introduction

Why have engineering disciplines been so productive? I believe it is because they quantitatively describe deterministic processes. Are they more than just an application of mathematics and science? What additional insight do they provide? Can those insights be applied to the realm of decision making? That is the core question addressed in this book.

I believe the answers to all of these questions are related to engineering's ability to extrapolate, in a useful and practical way, the behavior of events from the local to the global both in space and in time. It is taking the qualitative to the quantitative. In other words, engineering makes quantitative and "correct" predictions from a limited starting point! It accomplishes this using the scientific method along with the calculus-based mathematics of Newton that makes concrete the continuous and topological interconnection of events.

The scientific method provides a common base to collect, understand and organize the phenomena around a theoretical framework. The method provides a basis for reconciling failures of understanding and feeding that back so the framework can be improved. This framework has evolved to the point where it can make extrapolations from the local to the global using accurate data based on detailed measurements. The extrapolation is based on rules that have been vetted by historical precedent. Thus engineering has codified how things are put together, how long and whether they hold their shape and how changes occur over time. In other words engineering is based on attributes of forms that are both persistent over time and space, as well as on attributes that exhibit organized variability over time. The strength of engineering is a practical encapsulation of algebra, topology and geometry, whose power and reach in mathematics and science has been long argued (Bourbaki, 1968).

I believe that it is possible to apply the engineering discipline to decision making processes, despite knowing the reluctance of many to accept the possibility that human processes are described by the same tools of science that describe physical processes. I believe there is a solid scientific case that decision processes are deterministic. Decisions are actions that occur in the physical world. These actions occur in response to forces without need of postulating any other agents. What may not be apparent is how the reasoning from the physical sciences deals with human complexity. Yet many have already applied such reasoning in game theory and economic behavior, as for example those following the work of (Von Neumann & Morgenstern, 1944).

You may counter however that game theory rests on a static view of the world. Predictions about future behavior are based on an understanding of random events using Bayesian probability theory (Bayes, 1764). Recent work (Barabási, 2003) however on networks takes an opposite view. First of all networks provide the topological view of whatever underlying mechanisms might exist. The recent advances indicate that the networks are not random but behave according to laws of nature that are not unlike those in the physical sciences. It strongly supports the idea that behaviors are not random but deterministic. Though that work focuses on networks it provides support for our deterministic approach using the ideas of differential geometry. Indeed, the theory of algebraic topology (Eilenberg & Steenrod, 1952) provides details of how algebraic topological properties (graphs and networks) are exhibited in topological theories. Such network examples from the World Wide Web provide wonderful examples of why the approach taken here might have relevance to the real world.

Our view is that physical theories generalize from static theories to dynamic theories without resorting to the Bayesian stochastic approach. In electrical engineering we move from DC circuits with static behaviors to AC circuits with steady-state behavior, for example. The Bayesian approach is replaced by the physical principle of least action. In volume 1, this approach was applied to decisions (Thomas G. H., 2006), where we assigned a natural metric between decision events, related to the game theory notion of payoffs. The resultant behavior is analogous to the steady-state behaviors of AC circuits. To be more precise, we found behaviors that were analogous to resonant AC circuits in which only a few discrete frequencies contribute. The decisions follow the dynamics based on the metric of the space if we again start with a least action principle.

In this book we extend these results using numerical solutions to the exact equations of the proposed ***decision process theory***[1], along with the tools necessary for generating such solutions. We extend the notion that decisions follow circular (vortical) paths rather than being at fixed equilibrium positions. We address the question of how to obtain a more extensive set of exact solutions. We also extend the development of the theory to provide a sound pedagogical basis of these ideas for readers without a background in game theory, physics or differential geometry. In particular, we look at decisions as an engineering problem, one in which the decision process exhibits a ***characteristic behavior*** that is intrinsic to the process, as well as a ***forced behavior*** set by the initial conditions. This is in exact analogy to an electrical engineering network with an applied load being forced by a source. There too we see both a resonant or characteristic behavior (discrete frequencies) along with a forced behavior (arbitrary frequencies). We too find it useful to study such behaviors by considering the ***harmonic steady-state waves*** of the system, just as it is useful in electrical engineering to use phasor harmonics.

The concept of ***stationary flow*** is one in which *characteristic* and *forced* vortical behaviors play a central role. Equivalently we call such behaviors ***steady-state flow***. There is also a close atmospheric physics analogy to these equations, namely the equations describing the fluid flow of the atmosphere in which jet streams, tornadoes and hurricanes provide a distinct aspect of the observed weather. They may in fact be stationary, yet they are not simple extensions of linear local behaviors: such global behaviors such as turbulence are not visible when examining only the local behaviors.

We address the central thesis of the theory, that decisions are deterministic, in three parts. In the first part, chapters 1-4, we lay the theoretical foundations starting with the descriptive basis of game theory, which we also take as our descriptive basis. In the second part, chapters 5-7, we inquire into the nature of the theory with the scenarios for the prisoner's dilemma and Robinson Crusoe economics and conclude with a general review of game theory and economic approaches. In the third part, chapters 8-12, we numerically solve games with more than one person, each of which can have one or more strategies. These solutions elucidate the engineering issues that arise when analyzing decision processes based on our deterministic perspective. This third part illustrates the AC behaviors and shows the reader how the theory can be applied in a variety of situations.

As an introduction (chapter 1) to part 1, we lay out enough of an economic foundation to demonstrate the type of interesting economic problems we might study, such as the prisoner's dilemma. We also provide the physical foundation for the theory. In later chapters, we demonstrate complete solutions to such problems and find that our results suggest answers to dilemmas that have been noted in the literature. For example, our solution of the prisoner's dilemma contradicts the game theory intuition that the prisoners would be best off if they cooperated with the authorities and confessed. We will be able to demonstrate under what conditions this intuition is in fact supported, providing an example in which we gain new insight over past theories.

We put into context that a large number of problems in game theory are two person games in which each player has two strategic choices. Such problems are completely solved in game theory and are also completely solved for the dynamic theory, which we demonstrate in later chapters. In this chapter, we provide the context for economic situations with any number of players and any number of strategic choices.

In order to find solutions to such problems, we look at equations from differential geometry in the same way as electrical engineers. For example, we see that our equations are similar to equations in general relativity, so our approach of *harmonics* taken from electrical engineering (*phasors*) may be unorthodox to even physicists and we think a novel approach to solving equations from general relativity. Thus an introduction to physics is essential. Therefore in this chapter we provide a brief survey of the

[1] We indicate by bold-italics distinctions that we believe are new ways to speak about decision processes.

relevant ideas starting from Newtonian mechanics and beyond, including the theories of Maxwell. The latter theory is particularly relevant to our decision process theory.

The challenge of describing a physical deterministic theory is addressed using the language of differential geometry. A side benefit of such a general mathematical approach is that it suggests new distinctions about familiar phenomena. In this chapter we introduce some of the modern terminology and distinctions associated with differential geometry. This language may not be familiar to those whose expertise is decision making, to those familiar with economic thoughts or those familiar with game theory. Of course, the ideas of game theory were not generally familiar either at the outset of that theory.

As a possible motivation for readers who may not yet see the relevance, consider some of the ideas in the current press that we might like to understand better. Based on our work here, we distinguish between decisions made based on a sense of ***entitlement*** from those based on a sense of ***engagement*** with others. We will show that they follow different mechanisms and both contribute to all decision processes. We might equate *entitlement* to decisions made by our gut and *engagement* with decisions made by our reflective sense. To understand such distinctions we need to delve more deeply into the theory.

To meet the challenge of using the language of differential geometry for economic processes, we follow what we believe is a sound pedagogical approach and treat the problem in the same way as we would treat the training of engineering students. With this in mind we have selected appropriate ideas from differential geometry, electromagnetic theory and gauge theory including the theory of gravitation. We have also selected the elementary ideas of game theory and value theory. We believe it possible for a student who masters these ideas to have sufficient knowledge to understand and apply the ideas to practical problems. Indeed, we hope that students will have sufficient understanding to extend these ideas.

Here are the key ideas presented in this chapter. We present our theory, ***decision process theory*** following the techniques of Lagrange (for the expert, this indicates what we intend; for the non-expert we will say in sufficient detail what we mean so that you can use the concepts) and argue that it is useful to break forces into those that are ***applied*** and those that are ***constraints***, allowing the number of variables to be drastically reduced. These techniques relate to the key physical principle, the principle of least action that we use to create a causal theory. Without losing any generality, we can describe processes using variables that are *applied* and matched to the problem. For decisions we suggest that the key variables are the strategies available to each participant in the decision.

Once this is established, we bring together the key concepts of decision making (such as strategies and utility) and show that they can be described by calculus-based rules that allow the extrapolation from the local to the global, both in time and space. The extrapolation relies on detailed local measurements from which one gains insight into the global attributes of the decision process.

The key aspects of this process are firstly ***normative***, which we borrow from game theory: every decision can be put into a standard or ***normal form*** so that its aspects can be measured. Secondly, every decision involves choices some of which are ***persistent***, unchanging and ***inactive***, the rest are ***variable*** and ***active***. Thirdly, the decision processes are either the ***local*** processes that we deal with so successfully or the global processes that we deal with less well and often lead to unwanted results. These attributes can be put together in a common ***decision process theory*** (Thomas G. H., 2006), in which individual choices result in collective decision behaviors that are ***vortex-inducing.*** We describe in substantial detail how to extend and apply this theory to decision processes from both a theoretical position and a practical and computational position. Our goal is to explore in depth the underlying mechanisms behind these *vortex-inducing* behaviors.

In approaching this chapter and in fact this book, it is worth emphasizing that our approach utilizes the scientific method, which is more than just a discipline that deals with measurements and mathematics. It provides a common ground for discussion so that the ideas become public rather than private. Arguments can be carried out between people with substantial disagreements with a reasonable chance for a successful resolution. It is not a discipline where theories are chosen based on individual desires. This common ground approach has proved successful in science and we extend it to decision making. We must therefore show that the theory is a framework for all decision making discussions without exception. We thus propose more than a mathematical model or a narrow theoretical set of ideas that are fine tuned to describe a single phenomenon. We propose a systematic way of approaching a body of phenomena: which lies first in the physical domain no less than the phenomena of planetary orbits; which lies second in the biological domain of physical sciences such as molecular biology; and which lies third in the social and psychological domain that we assert is also governed by scientific laws. The theory must therefore be of local applicability and global scope. In short it must be a scientific theory consistent with the physical sciences.

It is within this context that we approach decision engineering as a calculus-based discipline, which could be used as the basis for a course. A necessary prerequisite for this course is the same as an upper division course in mechanical engineering, electrical engineering or bio-engineering or the equivalent. Typically an engineer is required to take two years of calculus and one year of the physical sciences and sufficient courses in each of the engineering disciplines to have a balanced view of engineering. For this book, we impose the same requirement. It is desirable in addition that the students have taken a course in electromagnetic fields and a business course in the theory of games.

The core course might be taught in three 10-week quarters or two semesters. We envision that the first quarter session of the course would cover chapters 1-3, the conceptual theoretical foundations of ***decision process theory***. The foundations are based strongly on an electrical engineer's view from the 21st Century modified to include the post-Newtonian formulation of mechanics by Einstein and a modern understanding of fields from quantum mechanics that replaces classical probability ideas.

The second session of the course would cover chapter 4, the theoretical framework and working models of ***decision process theory***. This session provides the technical essentials and can be covered in great detail (graduate level course) or as an overview (undergraduate level course). It would also cover chapters 5-7, including detailed and completely solved examples of the Prisoner's Dilemma (chapter 5), Robinson Crusoe scenarios (chapter 6), and critical discussions of the basis of the theory (chapter 7).

The third session would include chapters 8-9 on how to apply and extend the *decision process theory* to realistic applications starting with general two-person games (chapters 8-9). We suggest that the numerical work be obtained using the Mathematica© software by (Wolfram, 1992), which we have utilized throughout the book. The third session would return and review (chapter 7) key concepts in the literature and in the process take on a new perspective that game theory provides a set of ***known behaviors*** and ***persistent behaviors*** (chapter 8). The latter we associate with ***code of conduct***; which resolves the prisoner's dilemma and changes the foundational assumptions of our *game theory* starting point. The third session would conclude with an exploration of stable structures that are consequences of *decision process theory* (chapter 11), which highlight the global aspects of the theory and provide a starting point for future research.

As a guide, each chapter contains a specification of the outcomes expected for the student, an exposition of the ideas of that chapter and a list of problems or exercises that take the student further. Though it is desirable for the student to have the engineering and business prerequisites, every attempt is made to make the book self-contained.

We start with the prerequisites from the theory of games followed by mathematical and engineering prerequisites that form the conceptual basis for this new discipline.

1.1 ***Foundational game theory***

In order to make the conceptual basis for cause and effect relevant to decisions, we frame decisions as events that occur in some space at a moment in time. Here we review attempts to do this and find that foundational aspects of ***game theory*** provide the perfect vehicle in which to express our ***decision process theory***.

To provide a roadmap of where we want to go and what we need, we start with a preliminary view of our *decision process theory* and its relationship to engineering and to mathematical and physical theories. The foundational concepts from *game theory* that we find appealing for our ***decision process theory*** are that every decision process can be represented in a standard way and put into a ***normal form***, an insight of (Von Neumann & Morgenstern, 1944). They prove this under very general considerations, creating a theory of games and economic behavior, where games in their context go beyond recreational games to include real world economic behaviors. They envisioned that this way of looking at decision processes provides the basis for a *quantitative* way of dealing with social phenomena. See for example (Myerson, 1991), for an undergraduate course on game theory and (Mas-Colell, Whinston, & Green, 1995), for an undergraduate course on microeconomics.

We note and will have reason to refer to other work, such as (Nash, 1951), who improved and extended game theory and who also focused on the equilibrium and static attributes of the decision process. If we think of choices that change in time as flows, then we can say that these theorists characterize the flows as being the static remnants after all transient effects have died out. These theorists actually go further in that they assume that the flows are the same everywhere; the flows are equilibrium values akin to DC currents in a circuit.

In our view, in order to address questions such as recessions, cost over-runs and personal bankruptcies, we need to understand flows akin to AC currents as well as the transient behaviors; we need to understand the way decision processes depend on time. In this sense our approach diverges from much of the equilibrium focused literature. We should share some common ground with the time based theories and models. We return to this in chapter 7, and provide the historical context.

In game theory, the normal form consists of ***pure strategies*** that represent the full set of choices that need to be made for a game (decision). Thus in a chess game, in principle you could lay out what ***move*** you would make for every possible move your opponent might make for the duration of the game. You would need an enormously long piece of paper! The paper would capture a decision tree, with each possible path down the tree representing a pure strategy. Your opponent would make a corresponding list. The ***play*** of the game would consist of each person picking one of the pure strategies. Game theory recognizes however that when a game is played many times, there is a disadvantage to one side if the other side can predict what choice you might make. There is an advantage of playing a mix of strategies. In the static context of game theory, it is envisioned that all transients have died out and the player will have learned to pick a particular ***mixed strategy*** by choosing the pure strategies with frequencies or weights that are known. Under reasonable conditions, a game then has an equilibrium mixed strategy for each player of the game in the sense that no player can do better than that mixed strategy. At this point we go no further with game theory since we need only their concept of pure strategies and a modified view of their mixed strategies that we describe below. However, we note that our theory must include game theory as a special case of equilibrium behavior. Still, we must develop the framework in more detail before attempting to understand what it might mean to have an equilibrium strategy or what "doing better" implies.

In creating a new *decision process theory* using the foundational ideas from *game theory*, we make two additional assumptions. First, we view mixed strategies not as fundamentally random choices, but as the frequencies of successive decisions averaged over a sufficiently small sampling time. This is consistent with the original assumptions of the theory (Von Neumann & Morgenstern, 1944), though not with much of the more recent literature (see chapter 7). The choices may or may not be made randomly. In general every decision is the choice of a mixed strategy.

Second, we consider decisions as social phenomena carried out by many ***players***, each making what they consider to be independent decisions. They are aware however, not only of the outcome for themselves, but have their own view of the outcomes of their neighbors. We believe it is plausible that decisions made in one strategic part of space at one moment of time will therefore propagate not only forward in time but outward from this point. The characteristics of the propagation should be specified by the theory. This is analogous to the specification of the wave properties of a lake or sea given knowledge of some initial disturbance at some point and at some time. In the next two subsections we make this analogy more concrete.

1.2 *Newtonian mechanics*

The physical world consists of a variety of phenomena including decisions. It is helpful to review how classical Newtonian mechanics deals with physical phenomena, which we do in this subsection. We are particularly interested in how classical theory deals with phenomena in which some aspects are of interest and others must be accounted for but the details are not of interest, such as constraints. We thus deal with how Newtonian mechanics deals with complexity including, we argue, how we might deal with decisions.

Newton's famous equation $F = ma$ summarizes three separate laws. First it says that a body at rest stays at rest. Second it says that bodies that move with constant velocity (zero acceleration) continue to do so if they are not subject to an outside force. Third it says that for every action, there is an equal and opposite reaction. In other words any imbalance in the forces results in acceleration. Newton's equation depends on the second order rate of change of the position $\mathbf{r} = \begin{pmatrix} x & y & z \end{pmatrix}$ in terms of the x, y and z coordinates and the vector $\mathbf{F} = \begin{pmatrix} F_x & F_y & F_z \end{pmatrix}$ that represents the force along each of these directions respectively:

$$\mathbf{F} = m\ddot{\mathbf{r}} \tag{1.1}$$

We represent the rate of change using a "single dot" so that $\dot{\mathbf{r}}$ is the velocity and the "double dots" $\ddot{\mathbf{r}}$ represents the second order rate of change, namely the rate of change of the velocity, the acceleration. Newtonian physics reduces every problem, no matter how complex, to the use of this fundamental rule for a single body, where the proportionality constant is the ***mass*** of the body. The mass represents the inertia of a body. It is an observational fact that a massive body accelerates more slowly from rest than a light body. Mass has an experimental basis—it can be measured in units of the mass of smaller bodies—yet it does not have a clear origin other than the above equation. Inertia is a property of complex systems as well as simple, since the complex systems are made from the simple ones.

We rarely deal with a single ("rigid") body in physics. The simplest interesting case is two bodies attracted to each other by a gravitational force. Newton observed that the force is proportional to the masses of each body and inversely proportional to the square of their distances. In this way he could show that the theory would agree with the observations of Kepler that equal areas would be swept out in equal times.

A success of Newtonian mechanics is the accuracy of the predictions that result from these two statements. A success of the scientific method on which the theory rests is that as challenges to the theory have occurred, the framework for discussion has remained or evolved in a systematic way. For example it was discovered by Einstein that these two examples contain a fatal flaw for Newtonian mechanics, since they implicitly assume that the influence of one body is felt instantaneously by the other body. More careful observations show that action at a distance is not instantaneous, but is an effect that travels with a finite speed. A related flaw is the notion of a rigid body, which again would require the influence to travel instantaneously.

Despite the successes for the motion of two bodies under gravitational influence, such as the motion of the earth around the Sun and the belief that the laws hold generally, the Newtonian *formulation* as it stands does not deal *directly* with other real world situations. A car moving along the highway might be idealized as a single object, but the road on which the car moves is an important contributor to the forces acting on the car. Using Newton's formulation, you must account for all the forces, including those keeping the car from sinking into the pavement. Calculations that include these effects are long and once complete, have information that is not normally of great interest, such as knowing the value of the pavement force. It is at this point that we see a connection to decision theory. Here too we have many forces to deal with, but have a suspicion that their inclusion will yield information that may not be of great interest.

Being able to formulate problems in a more effective way was addressed by many researchers over the years and centuries after Newton. Their starting point was to write out all of Newton's equations, one for each body in the problem. This means idealizing in some cases a body as being made up of infinitesimally small parts. For the car, this would idealize the highway as being made up of lots of small bits, each characterized by a mass, acceleration and the net force acting on that bit:

$$\mathbf{F}_k = m_k \ddot{\mathbf{r}}_k \tag{1.2}$$

This represents many (second order differential) equations and takes into account all the forces. The forces of interest are *applied* and *external* (the motive force moving the car forward) and the forces that are not of interest are *constraints* or *internal* (the gravitational forces holding the car to the road and the atomic forces preventing the car from sinking into the road as well as those holding the bits of the car together). These definitions are not absolute, but relative to the interest of the person solving the problem.

A rigid body is held together by internal forces and such forces normally do no *work* (see below). A formulation that even partially excludes such internal constraining forces is advantageous. To appreciate this approach, it is helpful to introduce the concept of ***work***, which is defined as an integral made up of the sum of the product of the force times incremental distances along a specific path. Work has the dimensions of energy. Newtonian mechanics distinguishes energy associated with motion as kinetic $\frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ defined as one-half the product of the mass and the sum of the squares of the velocity components. Newtonian mechanics distinguishes energy associated with *forces,* as potential energy: typically the variation of the potential energy with distance along a given axis determines the force along that axis. In both cases energy has dimensions of a force times a distance. For so-called conservative systems, a consequence of Newtonian mechanics is that the sum of the kinetic and potential energy is constant. This is the case for two bodies interacting through the gravitational field. The potential energy in this case is proportional to the mass and inversely proportional to the distance. So how does this help us?

1.3 *Newtonian mechanics, Lagrange and game theory*

The idea is to formulate mechanics in a way that we can apply the general ideas to systems that are complex, such as ***decision process theory***. Such a formulation is Lagrange's, which starts with Eq. (1.2) and derives a new equivalent set of equations in which only the active and external forces are present. In doing so, he provides a general way of formulating mechanics that has proved useful even in areas where Newtonian mechanics breaks down as suggested by Einstein. In the Lagrange formulation, first we consider not the acceleration as given by the double rate of change of the coordinate, but consider the acceleration in terms of the momentum defined as $\mathbf{p}_k = m_k \dot{\mathbf{r}}_k$, the product of the mass and the velocity. The momentum is first order in the rate of change of the coordinate (the "velocity") and the force is first order in the rate of change of the momentum:

$$\mathbf{F}_k = \dot{\mathbf{p}}_k \tag{1.3}$$

Newton's equations are second order in the derivative, whereas Lagrange separates them into two sets of coupled first order equations. This may not seem like an improvement, but mathematicians have found this effective, so bear with us. They often use this device when studying differential equations of arbitrary order: they reduce the equations to a set of first order equations, where they have created a set of theorems for proving existence and uniqueness of solutions. For us, the more practical consequence is that we can often solve such equations numerically using application programs on our laptops. The general theory of solving such equations demonstrates that there will always be unique solutions when the initial value of each first order equation is specified. The real advantage however, is that we can more easily separate out the active from the constraint forces.

Lagrange attacks the key question of constraints with the goal of eliminating the variables that are of no interest. He considers the work done on the system if the coordinates undergo a small displacement [denoted by $\delta \mathbf{r}_k$]:

$$\sum_k \left(\mathbf{F}_k - \dot{\mathbf{p}}_k \right) \delta \mathbf{r}_k = 0 \tag{1.4}$$

There are applied forces and constraining forces. If the constraining forces do no work[2] and if the constraints are expressible as formulae involving only the coordinates $f\left(r_1, \cdots, r_N, t\right) = 0$ (called ***holonomic constraints***), then only the *applied* forces appear in the above formula:

[2] The constraint force does no work because the force moves through no distance.

$$\sum_k \left(\mathbf{F}^{(a)}{}_k - \dot{\mathbf{p}}_k\right)\delta\mathbf{r}_k = 0\,, \tag{1.5}$$

where $\mathbf{F}^{(a)}{}_k$ are the components of the applied force. The positions of the objects are not independent but constrained. The Lagrangian point of view is that any set of independent coordinates $\{q_\alpha\}$ (not necessarily equal to the above $\mathbf{r}_k$ coordinates) can be used to rewrite the above equation. Lagrange showed that the equations can be written in terms of the kinetic energy T now expressed in terms of the independent forces and the applied forces Q_α as they relate to these independent coordinates:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \tag{1.6}$$

The derivation of the above equation involves standard mathematical manipulations and relates these derived quantities to the original Newtonian quantities:

$$\begin{aligned} T &= \tfrac{1}{2}\sum m_k \dot{\mathbf{r}}_\mathbf{k}{}^2 \\ Q_\alpha &= \sum \mathbf{F}^{(\mathbf{a})}{}_\mathbf{k} \cdot \frac{\partial \mathbf{r}_\mathbf{k}}{\partial q_\alpha} \end{aligned} \tag{1.7}$$

The resultant Lagrange equations are an exact restatement of Newtonian mechanics involving only the active forces and the independent coordinates.

The subtle shift is away from the original set of all forces, to the set of applied forces that govern the motion of interest and the kinetic energy of the system including all the components, written in terms of an independent set of variables. We emphasize that this approach is really equivalent to the Newtonian approach as applied to the applied forces. It provides a solution for the constraint or internal forces in the special case that the special independent variables are taken to be the original coordinates, in which case it is assumed that the problem has been specified in its entirety with no constraints.

The expression obtained by Lagrange may also be written in terms of the kinetic and potential energy. For the important class of problems in which there is a potential energy V, the applied force Q_α is given in terms of the independent variables:

$$F^{(a)}{}_k = -\frac{\partial V}{\partial r_k} + \frac{d}{dt}\left(\frac{\partial V}{\partial v_k}\right) \Rightarrow Q_\alpha = -\frac{\partial V}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right) \tag{1.8}$$

In this case the equation of motion is expressed in terms of the difference between the kinetic and potential energy, called the ***Lagrangian*** $L = T - V$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \tag{1.9}$$

We see that all of the information about the problem is now reflected in the Lagrangian that has dimensions of energy. Further these equations of motion solve the problem posed, namely the equations involve only the applied coordinates, though all aspects of the kinetic and potential energy are taken into account. In the next chapter (section 2.1), we establish that the Lagrange equations of motion can be formulated in terms of a principle of least action.

In the case of a car moving along a highway, though the motion occurs in three dimensions, the configuration space needed to describe the situations is 3^N where N is a truly large number representing the number of bodies interacting (which is of the order 10^{24}). Thus the practical application recognizes that the important question is just the motion along the two-dimensional highway (or indeed just a single dimension if the car stays in its lane). The above view demonstrates that there is no loss of generality focusing on the two dimensional coordinates in determining the motion as long as the kinetic and potential energies represent their totals faithfully.

It is therefore plausible that we deal with decisions in the same way. The foundational aspects of *game theory* give us the applied strategies that we should consider, though there may be other variables to consider. The applied strategies are the generalized coordinates of the Lagrange approach. As long as we faithfully include all aspects of the kinetic and potential energy, we have a specification that takes all the applied aspects of the problem into account. We take this view and apply it to our conceptual basis from foundational game theory, section 1.1.

1.4 *Conceptual basis and payoffs*

If we take as our conceptual basis for decisions only the foundational aspects of game theory, we make little if any use of the dynamic content that has been unearthed by several decades of work on the ideas of (Von Neumann & Morgenstern, 1944) and (Nash, 1951). It is our view that the scientific method has power in its ability to incorporate ideas and that there are many game theory ideas that need to be incorporated into any theory for decision processes. Our analysis suggests that a common feature of game theory models is that decision processes are characterized by ***payoffs***. The outcome of a decision can be quantified numerically as a value each player receives (or pays) as a result of each player choosing a pure strategy. The payoffs for mixed strategy choices are determined from these pure strategy payoffs. We must find a way to incorporate payoffs into our *decision process theory.*

We believe therefore that we must incorporate two related concepts from game theory: the concept of a ***game value*** and the concept of an ***equilibrium*** set of strategies. Our ***decision process theory*** is not *derived* from game theory so we don't have a precise method for adding these concepts. Our method can be illustrated best by using an example and showing how it is treated in game theory and how we might then incorporate these game theory ideas into our framework. The example we have chosen is the Prisoner's Dilemma.

The prisoner's dilemma has been extensively investigated by game theorists, for example (Rapoport, 1989) and has been scrutinized in both theoretical and empirical contexts. The scenario is that two prisoners are being held for a crime where it is suspected they have acted in concert. Each is given a choice to confess or not confess with penalties that are supposed to induce confession of their guilt. However if neither confesses they will get off lightly. If both confess they will be penalized but not as severely as the case in which one confesses and implicates the other. The dilemma is that the prisoners would be collectively best off by not confessing. However the game theory analysis below recommends they both confess.

We start with the formulation of the prisoner's dilemma as a game in ***normal form*** between two players/prisoners specified by the following payoffs to player 1.

G^1_{12}	N_2	C_2
N_1	-0.1	-1
C_1	0	-0.9

(1.10)

There are identical payoffs to player 2:

G^2_{21}	N_1	C_1
N_2	-0.1	-1
C_2	0	-0.9

(1.11)

Game theory supposes that there are quantitative payoffs for the matrix $\mathbf{G}^k$ of possibilities as illustrated above for player $k \in \{1,2\}$. We describe the meaning of the elements of the matrix in detail for player 1, noting a similar description holds for player 2. For player 1, the rows are labeled (C_1) if player 1 confesses and (N_1) if player 1 does not confess. The columns are labeled in a similar manner (N_2) and (C_2). The payoffs reflect quantitatively the posed problem: if both confess player 1 loses 0.9 units, if both don't confess player 1 loses a much smaller value: 0.1. If player 1 confesses and implicates player 2 who does not confess, then player 1 loses 0.0 units. On the contrary, if player 1 does not confess and is implicated by player 2, player 1 loses 1 unit.

Though rather simple, the example illustrates several general properties of *game theory*. The payoffs for the two players need not add up to zero; when they do, (Von Neumann & Morgenstern, 1944) called them ***zero-sum games***. Zero-sum games have been analyzed in detail and have an equilibrium value that is argued as follows. Suppose the payoffs Eq. (1.10) for player 1 were a zero-sum game. The most conservative strategy for player 1 would be to determine the minimum outcome for each of its pure strategy choices. If player 1 chooses (N_1), the minimum case is -1. If player 1 chooses (C_1) the minimum is -0.9. The maximum of these two is -0.9. The most conservative strategy is to choose this max-min. There may not always be a max-min solution however. What (Von Neumann & Morgenstern, 1944) showed however, is that for a zero-sum game with any number of players, each with any number of strategies, there is always a max-min using mixed strategies. This is called an ***equilibrium strategy***. It is the best one can do acting defensively. When there is equilibrium, there is also a ***game value*** equal to the payoff associated with the equilibrium strategy (or weighted payoff if the strategies are mixed).

When the game is not zero-sum and there are separate payoffs for every player, the above argument has been replaced by (Nash, 1951). He showed that again there will be a best choice now called the ***Nash equilibrium***. Nash says that the best strategy has the property that no player can do better against this strategy as long as all the other players adhere to their best strategy. There will now be multiple *game values*, one for each player at the Nash equilibrium. By inspection, the Nash equilibrium for the prisoner's dilemma is that each player confesses. The *game value* for each player is -0.9.

Notwithstanding the great insight provided by the Nash equilibrium and the importance of considering the payoffs separately for each player, we believe insight into the dynamics comes first from looking at each zero-sum game and studying *how* one arrives at that equilibrium strategy. For the example chosen here, it is possible to determine that strategy by inspection. In the general case the max-min problem is solved numerically. One general method is to first convert the game to a fair game (one whose *game value* is zero) which is equivalent in that by construction it has the same equilibrium value. For player 1, the general method assumes that the fair game has one additional strategy, the hedge, whose payoff is adjusted to get the required equilibrium:

F^1_{ab}	N_2	C_2	N_1	C_1	H
N_2	0	0	0.1	0	0
C_2	0	0	1.0	0.9	$-\frac{1}{m}0.9$
N_1	-0.1	-1.0	0	0	$\frac{1}{m}1.0$
C_1	0	-0.9	0	0	$\frac{1}{m}0.9$
H	0	$\frac{1}{m}0.9$	$-\frac{1}{m}1.0$	$-\frac{1}{m}0.9$	0

(1.12)

This fair game produces exactly the same result as the original game.

However, finding the equilibrium value for this fair game is analogous to a dynamic process. It is of great interest to us that one of the methods of finding the equilibrium is to write a differential equation involving these payoffs of the form:

$$\mathbf{g}\cdot\frac{d\mathbf{V}}{d\tau} = -e\mathbf{F}\cdot\mathbf{V} \tag{1.13}$$

Though in a different form, this method was introduced by Von Neumann and is described in (Luce & Raiffa, 1957). The rate of change of the mixed strategy is proportional to the product of the payoff matrix and the mixed strategy. At equilibrium, the rate of change of the mixed strategy is zero so the product is also zero. The advantage of the reformulation of the game to a fair game is that the equilibrium is defined by the product vanishing. This technique can be done for any zero-sum game and any number of players with any number of strategies.

We believe that we now have a specific attribute to look for in a dynamic theory. The theory needs to provide in a natural way a payoff matrix of the type above that is associated with each player. The notions of equilibrium, Nash equilibrium and *game value* then have generalizations in the theory. The dynamics represented by Eq. (1.13) for a payoff matrix that is antisymmetric is analogous to electrodynamics. We present the arguments for why this is true and relevant in the next section.

1.5 *Electromagnetic fields as payoffs*

We find that a characteristic behavior of decision processes is a tornado or ***vortex***-like behavior such as one sees in weather phenomena (Thomas G. H., 2006). A fundamental goal of this book is to explore the important underlying deterministic mechanisms, one of which is *vorticity*. A key to that understanding is the similarity between the form of Maxwell's equations and the form we identified in the previous section for payoffs. However to see that similarity and appreciate its import, we have to rewrite the fundamental laws of Maxwell in the covariant form introduced by Einstein. We shall review the physical origins of Maxwell's laws and then show how the equations can be rewritten in the covariant form. We will then see the connection.

Some of you may recall that Maxwell based his work on phenomena that in part were known in antiquity. At the core are the ideas that there exists an attribute called ***charge*** that is associated with matter. The charge can be positive, negative or zero: like-charges repel and opposite-charges attract without any visible matter acting as the cause or intermediary. In Newtonian language the law of attraction is an instantaneous action-at-a-distance force inversely proportional to the square of the distance and directly proportional to each charge. It is customary to describe the law of attraction as a charged body interacting with a vector ***electric field*** $\mathbf{E}$ defined at each point of space. The force experienced by a charged body is then proportional to its charge and the electric field in which it finds itself. Like gravity, what is notable about the electric field is that it makes things move "instantaneously" without any observable connection.

The existence of magnetism has also been known since antiquity. The most notable application is the use of magnetic material as a compass for navigation. Again the effect of magnetism appears to be an action-at-a-distance force and it is customary to describe the force in terms of a magnetic field $\mathbf{B}$. There is no particular reason to suspect that the electric and magnetic fields or the corresponding forces are related. However, in Europe a number of experimenters learned more about electric and magnetic phenomena. They observed how charged bodies move in the presence of a magnetic field and noted that when charges accelerate they generate not only an electric field but a magnetic field. Maxwell synthesized these diverse observations into a comprehensive theory that bears his name. He provided equations that summarized the laws of how a charge density ρ produces an electric or magnetic field based on its motion $\mathbf{V}$ and how the magnetic and electric fields influence the motion of charged particles:

$$
\begin{aligned}
&\partial_x E_x + \partial_y E_y + \partial_z E_z = \rho \\
&\partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \\
&\partial_x E_y - \partial_y E_x = -\frac{1}{c}\frac{\partial B_z}{\partial t}, \quad \partial_y E_z - \partial_z E_y = -\frac{1}{c}\frac{\partial B_x}{\partial t}, \quad \partial_z E_x - \partial_x E_z = -\frac{1}{c}\frac{\partial B_y}{\partial t} \\
&\partial_x B_y - \partial_y B_x = \frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{1}{c}\rho V_z \qquad (1.14) \\
&\partial_y B_z - \partial_z B_y = \frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{1}{c}\rho V_x \\
&\partial_z B_x - \partial_x B_z = \frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{1}{c}\rho V_y
\end{aligned}
$$

The first equation generates the law of attraction for charges (Coulomb's Law); the second equation expresses the fact that there have never been any magnetic charges observed; the simplest magnet always comes as a dipole. The next three equations reflect Faraday's Law that changing magnetic fields induce electric currents and hence an electric field. The last three equations agree with the experimental Ampère's Law that moving charge generates a magnetic field. The last three equations however are not identical to Ampère's Law because of the presence of the "conduction current," the extra gradient of the electric field with time. There is no contradiction since the early experiments were done using steady state electric fields and the extra term would have been zero.

Maxwell realized that without the extra term, charge would not be conserved. With the extra term we get a continuity equation similar to the equation for conservation of mass:

$$\frac{d\rho}{dt} + \rho\partial_j V_j = 0 \tag{1.15}$$

We introduce the convention that for repeated indices j we sum over the possible values x, y, z .The conservation law agrees with the observation that charge does not spontaneously appear or disappear: its flow in or out of a box or cell in space results in an increase or decrease of the charge inside that cell. So, like mass, charge is conserved within cells.

Maxwell's equations are consistent for electric and magnetic fields that vary in time. Moreover, the electric and magnetic fields separately satisfy wave equations in free space whenever the charge density is zero $\rho = 0$:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \partial_j \partial_j \mathbf{E} = 0 \\ \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \partial_j \partial_j \mathbf{B} = 0 \end{aligned} \tag{1.16}$$

This is an astounding result since it says that a wave will propagate through space with the speed set by the constant c in the theory. This was verified by the generation of radio waves. The waves associated with free space are called ***electromagnetic radiation*** and include not only radio waves but light, microwave and x-ray radiation.

The presence of radiation fundamentally changes the conceptual nature of the law of attraction between two charges (Coulomb's Law). This is most dramatically demonstrated by looking at the electric field that is generated from the motion of a moving charge q with position specified by a unit vector $\mathbf{e_r}$ a distance $\mathbf{r}$ away (Feynman, Leighton, & Sands, 1963, p. I.28.2):

$$\mathbf{E} = \frac{q}{4\pi}\left(\frac{\mathbf{e_r}}{r^2} + r\frac{\partial}{\partial t}\left(\frac{\mathbf{e_r}}{r^2}\right)_x + \left(\frac{\partial^2 \mathbf{e_r}}{\partial t^2}\right)_x \right) \tag{1.17}$$

The first term is what one would expect if the force were instantaneous. In Maxwell's theory, however, the force is not instantaneous. The effect of charge propagates with the velocity c and so is not felt at a given point until it has propagated there. Thus the force one gets is based on the "retarded" distance of where the charged body was. The additional correction terms to Coulomb's law are based on doing that arithmetic correctly. The correction terms are not measurable until the speeds are comparable to the constant c .

Moreover, the equations make connection with a seemingly unrelated phenomenon: light. We have learned a good deal about light since Newton. We know light travels as a wave and we know further that under certain circumstances it behaves like a particle moving in straight lines. The above analysis construes light as a special case of electromagnetic radiation. The constant that appears in the equations is the speed of light (in a vacuum) and is approximately 186,000 miles/sec. For such speeds, it is not surprising that correction terms were not initially noticed. A speed of 50 miles/sec, which seems quite fast being 180,000 miles/hour is only 0.00027 the velocity of light. The correction terms above are actually of the order of this amount squared, since the coefficient of the corrections to first order is zero.

Maxwell's equations have been written in a greatly simplified notation since Maxwell. The electric and magnetic fields are now understood to be part of a single tensor structure in space-time. We note the simplifications that occur when the electric and magnetic fields are written as a single antisymmetric ***electromagnetic field*** F_{ab} where the indices a and b can take on the three spatial coordinates and one time coordinate (represented as $x_0 = ct$):

$$\begin{aligned} &F_{x0} = E_x \quad F_{y0} = E_y \quad F_{z0} = E_z \\ &F_{xy} = B_z \quad F_{yz} = B_x \quad F_{zx} = B_y \\ &F_{ba} = -F_{ab} \end{aligned} \tag{1.18}$$

Given this notation, Maxwell's equations are:

$$\begin{aligned} &-\partial_x F_{0x} - \partial_y F_{0y} - \partial_z F_{0z} = \rho \\ &\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = 0 \\ &\partial_x F_{y0} + \partial_y F_{0x} + \partial_0 F_{xy} = 0 \\ &\partial_y F_{z0} + \partial_z F_{0y} + \partial_0 F_{yz} = 0 \\ &\partial_z F_{x0} + \partial_x F_{0z} + \partial_0 F_{zx} = 0 \\ &\partial_0 F_{z0} - \partial_x F_{zx} - \partial_y F_{zy} = -\frac{1}{c}\rho V_z \\ &\partial_0 F_{x0} - \partial_y F_{xy} - \partial_z F_{xz} = -\frac{1}{c}\rho V_x \\ &\partial_0 F_{y0} - \partial_z F_{yz} - \partial_x F_{yx} = -\frac{1}{c}\rho V_y \end{aligned} \tag{1.19}$$

There are two classes of equations. The first equation (Coulomb's Law) and the last three equations (Ampère's Law as modified by Maxwell) describe how charged matter provides the source for the electromagnetic field. The second equation (no magnetic monopoles) and the following three (Faraday's laws) have the generic form:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{1.20}$$

An analysis of this set of equation implies that the electromagnetic field F_{ab} is always determined in terms of a vector valued function A_a called the vector potential (in analogy to the potential energy):

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{1.21}$$

In order to write the Maxwell's equations more compactly (and thus make connection with the payoffs from the previous section), we have to take into account the fact that space coordinates have an extra minus sign compared to the time component x_0.

Maxwell's equations demonstrate transformation properties that are not the same as Newtonian mechanics. Ampère's Law, consisting of the last three equations of Eq. (1.19), Coulomb's Law, the first equation of Eq. (1.19), the conservation of charge Eq. (1.15) and the wave equations Eq. (1.16) reflect simple geometric forms when expressed in terms of the vector coordinate displacements $\{dx^0 = cdt, \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz\}$ and a symmetric "metric" tensor $g_{ab} = g_{ba}$ that has the values:

$$\begin{aligned} &g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \\ &a \neq b \Rightarrow g_{ab} = 0 \end{aligned} \tag{1.22}$$

We use g^{ab} as the inverse of the tensor (considered as a matrix), which in this case will have the same numerical values. We then distinguish different tensors depending on whether their indices are up or down by whether we have multiplied by this metric: $F_a^{\ b} = g^{bc} F_{ac}$.

In Maxwell's theory, light travels with a constant velocity c so that the distance light travels in an interval of time is dx^0 and so the following invariant length is zero:

$$ds^2 = dx^a g_{ab} dx^b = \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 \tag{1.23}$$

A general path in space and time whose length is given by an expression of this type describes non-Euclidean geometries (the geometry of curved spaces) that have been exhaustively studied by mathematicians, notable Riemann and used by Einstein as the basis for the theory of relativity. Using this notation we write the four equations that involve a source (Ampère's Law and Coulomb's Law) in a form that shows that all four equations are of the same form, the laws are ***covariant*** when expressed in terms of the four dimensions of space and time:

$$\partial_c F_{ab} g^{bc} = \frac{1}{c} \rho g_{ab} V^b \Leftrightarrow \partial_b F^{ab} = \frac{1}{c} \rho V^a \tag{1.24}$$

We introduced for the time component of the flow, $V^0 = 1$.

To complete the comparison of the electromagnetic fields to payoffs Eq. (1.13) we need the behavior of charged matter in an electromagnetic field, the Lorentz force:

$$\begin{aligned} m\frac{dV_x}{d\tau} &= eE_x + e\left(V_y B_z - V_z B_y\right) \\ m\frac{dV_y}{d\tau} &= eE_y + e\left(V_z B_x - V_x B_z\right) \\ m\frac{dV_z}{d\tau} &= eE_z + e\left(V_x B_y - V_y B_x\right) \end{aligned} \tag{1.25}$$

We put this into the covariant form, which also adds a time component:

$$m\frac{dV^a}{d\tau} = -eF^a{}_b V^b \tag{1.26}$$

Based on the comparison we see that the form of the payoff (Eq. (1.13) is the same as this (Lorentz force) form for the behavior of charged matter in an electromagnetic field. The comparison suggests we identify the hedge strategy with time, the flow with the mixed strategy flow and the payoff matrix with the electromagnetic field. There are differences however that we must still deal with. The dimensionality above is three space-dimensions and one time-dimension: the payoff matrix has a dimension $n+1$ based on the number n of pure strategies. Distances and energy are appropriate to decisions not physical distances. Both are a consequence of a differential geometry. This identification is helpful and we deal with these issues next.

1.6 *Units, payoffs and geometric forms*

Before embarking on a more complete justification of our ***decision process theory*** (section 1.7), we anticipate some of the results and address the issues raised in the last section. We suggest units for a theory of decisions and provide a generalization of electromagnetic theory to N space dimensions and 1 time dimension. This section is technical.

We will address the question of units in this expanded space first. For decision processes, we believe the appropriate units are time, effort and utility. They correspond to time, distance and energy, units we could easily adopt in the physical theory. One objection to our approach might be that we live in a 3+1 dimensional world. How can it be that there are more dimensions? Our provisional answer is that we are describing the behavior of multiple agents, each of which occupies space at any moment of time. Because of this many-body nature, we must allow for many dimensions. In a decision process they manifest as different strategic possibilities. It is thus plausible that we have many. Recall the car example (section 1.3). There in fact might be many more, but we are pulling out just those that are unconstrained and are of interest.

We use units of time $[T]$ that are big enough to sample enough decisions for an ongoing set of decisions to provide an accurate measure. This is not unlike picking units of distance in physics that allow for repeated measurements so that one can get an accurate measurement of distance. The idea is that the intervals can't be too big because there is no accuracy. They also can't be too small because the decision process may not be complete. These concepts are closely related to the *principle of least action* (section 2.6) and the concept of a minimal or quantum of action (section 2.7). In these sections we are able to sharpen the domain of validity of the *principle of least action* and the concept of time being "big enough".

Making a decision involves the ***act*** of making a choice and making the relevant effort in (distance) units $[L]$ to execute that choice. In a business, we might measure the effort in ***staff-years***. We could also measure the effort in ***staff-seconds*** if the decision process is sufficiently simple. The effort made per unit time is then the speed, which is given by the units of effort and time, $[L/T]$. For much of what follows, we pick the maximum possible speed c to be unity: in other words it is the maximum or ideal effort a member of the staff of the decision process might make. All other speeds will by definition be less than this ideal.

Distinct from the effort we make in choosing, there is the value or ***utility*** of that choice. We discuss *utility theory* in more detail in section 7.4. For now, it is sufficient to note that we measure *utility* as a value $[U]$ that is a convertible value such as dollars, pounds, Euros or yen. We are aware of the arguments in the literature that the value is only relative to the decision agent. We support aspects of this view. In section 7.7 however, we make the case that all utilities are convertible, though the mechanism may be complex. That being the case, we choose units for payoffs that are related to utility.

We start with the appropriate payoff potential A_a, which we take to be unit-less, in which case the payoff field F_{ab} has units of inverse effort, $\left[L^{-1}\right]$. To convert effort to utility, we require a ***productivity factor*** $1/\kappa$. We measure payoffs $\frac{1}{\kappa}F_{ab}$ as the utility per unit ***strategic surface***, $\left[U/L^{N-1}\right]$ where the strategic surface is a cross-section of the ***strategic volume*** of the N strategic choices of the decision process. Consequently, the *productivity factor* $1/\kappa$ has units $\left[U/L^{N-2}\right]$, which is utility per unit ***strategic boundary*** and will be consistent with our usage in Eq. (2.31) and subsequent usages such as Eq. (3.14).

We restate the basic equations from section 1.5 for the electromagnetic fields in the language of relativity and geometry by going from the differential formulation to an integral formulation, (Warner, 1971). We use the language of forms from differential geometry to represent the integral quantities of interest. We start with a ***0-form***, which is a scalar quantity φ. It has the property that its differential, integrated along *any* closed path is zero:

$$\oint d\varphi = 0 \tag{1.27}$$

The differential of a scalar depends on the gradient components:

$$d\varphi = \partial_a \varphi dx^a \tag{1.28}$$

It makes sense to think of the differentials dx^a as a basis (in the sense of a linear vector space) and the components along this basis $\partial_a \varphi$ as defining a new quantity called a ***1-form***. More generally, any set of components A_a define a 1-form, though in general the integral of this 1-form around a closed path is not zero:

$$\oint A_a dx^a \neq 0 \tag{1.29}$$

The 1-form $\mathbf{A}$ represents a differentially short path segment of a curve in space:

$$\mathbf{A} \equiv A_a dx^a \tag{1.30}$$

Line integrals are constructed from the integration of such segments along a curve. They represent the ***circulation*** of A_a along this path.

The differential $d\mathbf{A}$ of a 1-form defines a 2-form and represents the movement of the entire segment, which spans an area. In three-dimensional vector calculus, a differential area is conveniently written in terms of the ***cross product*** of two differential vectors, one along the segment and the other along the transport path. In higher dimensions the *cross product* is generalized to the antisymmetric ***wedge product*** of two 1-forms, a 2-form, defined in terms of the measure for area:

$$\mathbf{A} \wedge \mathbf{B} = A_a dx^a \wedge B_b dx^b = A_a B_b dx^a \wedge dx^b = \tfrac{1}{2}\left(A_a B_b - B_a A_b\right) dx^a \wedge dx^b \tag{1.31}$$

In general, the integral of a 2-form around a closed surface is not zero; however, in the special case that the 2-form is the differential of a 1-form, the closed surface integral is zero. In the general case, we think of the 2-form as the flow through the surface.

We use ***n-forms*** to re-write the basic electromagnetic field equations in integral form in any dimension of space time. We identify the payoff field with the electromagnetic field F_{ab}, using the units set out at the start of this section. The key properties of the field equations depend on whether certain closed integrals are zero or non-zero.

To begin, we define the field 2-form $\mathbf{F}$ in terms of the field intensity F_{ab} and the element of surface $dx^a \wedge dx^b$ formed from the corresponding components. The field intensity represents a flow through this surface element: this is the flux (field intensity per unit area) that flows through this infinitesimal 2-surface:

$$\mathbf{F} = \tfrac{1}{2!} F_{ab} dx^a \wedge dx^b \tag{1.32}$$

These forms are ideal mathematical constructs to describe the physical notion of flow of flux. The units of the field 2-form is effort $[L]$ (e.g. staff-years).

The integral or sum of this 2-form over a closed surface is then given by Stokes' Law for the differential form of the flux integrated over the enclosed 3-volume $d\mathbf{F} = \tfrac{1}{2!} \partial_c F_{ab} dx^c \wedge dx^a \wedge dx^b$:

$$\oint_{\partial K} \mathbf{F} = \int_K d\mathbf{F} = 0 \tag{1.33}$$

We make use of two general properties of n-forms. First, ***Stokes' Law*** for any n-form $\mathbf{G}$ expresses the fact that the integral of the differential of $\mathbf{G}$ over any subspace K of an n-dimensional space is equal to the integral of $\mathbf{G}$ around the closed boundary ∂K of that subspace:

$$\oint_{\partial K} \mathbf{G} = \int_K d\mathbf{G} \tag{1.34}$$

Second, the differential of any n-form that is constructed from the differential of an $n-1$-form is zero:

$$dd = 0 \tag{1.35}$$

The electromagnetic flux is the differential of the electromagnetic potential, $\mathbf{F} = d\mathbf{A}$. This carries over into any number of dimensions and so works for the payoff field in particular.

Considering time and space components separately, there are two types of equations that result from Eq. (1.33). The first occurs if the bounding 2-surfaces contain only spatial components.

$$\oint_{\partial K} F_{mn} dx^m \wedge dx^n = 0 \tag{1.36}$$

This is the analog of no magnetic monopoles $\oint_{\partial K} B \bullet dS = 0$. The net flux through the closed boundary is zero. The implication is that the magnetic flux, defined as the spatial components F_{mn} of the ***field intensity***, cannot stop or start inside a volume, but flows continuously. The differential form of this is $\partial_l F_{mn} + \partial_m F_{nl} + \partial_n F_{lm} = 0$.

The general form involves both space and time components:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{1.37}$$

The flux doesn't get created or destroyed inside a surface in which one of the dimensions is time. We prefer not to think of the time components in this way; rather we express the differential form in terms of the electric fields $E_m = F_{m0}$ and the magnetic fields which are the components F_{mn}:

$$\partial_0 F_{mn} + \partial_m F_{n0} + \partial_n F_{0m} = 0 \Rightarrow \partial_m E_n - \partial_n E_m = -\partial_0 F_{mn} \tag{1.38}$$

This is the generalization of ***Faraday's Law***, Eq. (1.20), showing that an electromotive force (left hand side) is generated whenever there is a magnetic flux that changes in time. The electromotive force is determined by the one dimensional integral around a closed curve of the magnetic flux that goes through the surface bounded by this curve (Faraday's law):

$$\oint_{\partial K} F_{m0}dx^m = -\frac{\partial}{\partial t}\int_K \tfrac{1}{2!}F_{mn}dx^m \wedge dx^n \qquad (1.39)$$

The left hand side is the *circulation* of the electric flux around a one dimensional boundary, and is defined as the *electric potential*. This equation is a special case of Eq. (1.33) where one of the dimensions is time and the initial and final times are very close. In this case the two dimensional surface on the left originates from Δt times a one-dimensional integral of the boundary, and the three dimensional volume on the right originates from the difference of two-dimensional volumes over a short interval of time. The above result is obtained by taking the time difference and dividing by the differential time. We think of the result as the time rate of change of the spatial components of the payoff flux flowing through a two-dimensional surface determining the electric potential.

In the general case of a payoff field, the ***magnetic flux*** (calling the space components of the field intensity F_{mn} the magnetic field) flows like a continuous media without sources or sinks (lines are not created or destroyed). Changes in the magnetic flux with time generates the electromotive force (unit-less) described by the left hand side of Eq.(1.39). The magnetic flux lines cause charged particles to move in patterns identical to the paths predicted by elementary game theory, identifying the magnetic field components with the payoff matrices of the players. In the stationary case, no electromotive forces are generated, analogous to the situation of stationary electric circuits: The above equation reduces to Kirchhoff's Law of electrostatics. The result of changing flux with time creates electrical forces that induce currents (motion of charges). Inertial energy is converted to electromagnetic energy and vice versa.

The remaining set of equations of Maxwell involves the sources of the electromagnetic fields. The sources are current 1-forms $\mathbf{j}$ constructed from the *circulation* of charges (utility flow j_a) **along** a path in space-time. For any flow at each point, there is a dual description: the flow along a path can be described by the flow through the orthogonal or dual surface. The form of the current flowing **through** the dual surface is in terms of the element of volume, an N-form:

$$*\mathbf{j} = \tfrac{1}{N!}\varepsilon_{ab\cdots f}\, j^a dx^b \wedge \cdots \wedge dx^f \qquad (1.40)$$

The dimension of space time is $N+1$ and we write the current in terms of a totally antisymmetric tensor $\varepsilon_{ab\cdots f}$ with $N+1$-indices and a vector j^a representing the ***current density*** flowing through the volume (the dual surface). Typically the current density is related to the charge density ρ_{ch} and velocity flow V^a by $j^a = \rho_{ch}V^a$. We measure current density in units of $\left[U/L^N\right]$.

This process of constructing a new tensor using the totally antisymmetric tensor is quite general and the resulting tensor is called the ***dual tensor*** and the operation of creating the dual is indicated by an asterisk. Corresponding to the 2-form payoff flux flowing **through** a 2-surface, there is a corresponding $(N-1)$-form payoff flux $*\mathbf{F}$ describing the flow **along** the dual space orthogonal to the 2-surface. The remaining Maxwell's equations are expressed in terms of this dual 2-form:

$$*\mathbf{F} = \tfrac{1}{2!(N-1)!}\varepsilon_{a\cdots def}F^{ef}dx^a \wedge \cdots \wedge dx^d \qquad (1.41)$$

The ***scaled payoff*** $*\mathbf{F}/\kappa$ has dimensions of utility and represents the flow of ***utility flow*** **along** an $N-1$ dimensional ***strategic surface***. Utility can be created and destroyed since the current acts as the source or sink. This leads to generalizations of Coulomb and Ampère's law that relate the differential of the scaled payoff to the current and are expressed here as a single relation:

$$*\mathbf{j} = d*\mathbf{F}/\kappa \qquad (1.42)$$

Through Stokes' theorem Eq. (1.34), the integral form is:

$$\oint_{\partial K} *\mathbf{F}/\kappa = \int_K d*\mathbf{F}/\kappa = \int_K *\mathbf{j} \tag{1.43}$$

The right hand side describes the utility and the dual payoff $*\mathbf{F}$ on the left hand side describes the effort to produce that payoff. The inverse of the ***productivity factor*** κ is a measure of the effort required to produce one unit of utility.

Utility is created or destroyed through the flow of currents. The units of the current are $\left[U/L^N\right]$. As before, these equations represent different physical situations depending on whether the components of the integrand are purely spatial or not. If we suppose that on the right, the volume is purely spatial, then we find that the integral is the charge over the surrounding volume, $\int_K \frac{1}{N!}\varepsilon_{0m\cdots n} j^0 dx^m \wedge \cdots \wedge dx^n$. In this case, the left hand side represents surfaces which are purely space like, so one and only one of the components can be time-like. The left hand side is therefore the net electric field density that is normal to the $N-1$ dimensional *strategic surface* bounding the N dimensional *strategic volume*, $\oint_{\partial K} F^{0m}\mathbf{S_m} = \kappa Q_{encl}$. This is the integral form of ***Coulomb's Law*** $\partial_m F^{0m} = \kappa\rho_{ch}$, Eq. (1.19), where the surface integral $\mathbf{S_m} = \frac{1}{(N-1)!}\varepsilon_{0p\cdots qm} dx^p \wedge \cdots \wedge dx^q$ is the dual of a 1-form in the N dimensional strategic subspace.

The decomposition of a *payoff field* into its spatial and time components is frame dependent. In each frame, the space coordinates determine a volume; the spatial *payoff* component (***flux***) that flows through any two-dimensional *strategic surface* of that volume is *magnetic*. The space-time *flux* that flows through a $N-1$ dimensional *strategic surface* is *electric*. The net number of *flux* lines that emanate from this spatial volume equals the enclosed *utility*. This nomenclature however is unnecessarily tied to the physics analogy, though at times it is comforting and provides a useful mnemonic for looking up standard results. We introduce new terminology to emphasize we are not tied to that analogy.

We have said that the total *utility* ("charge") is determined by the net flow of the *electric* components of the *payoff field* through the strategic surface. This supports the idea that *utility* is the source of *value* as used in *game theory*: *utility* determines the *electric* field and *value* as used in *game theory* becomes the *electric field* in *decision process theory*. We therefore suggest the *electric* decomposition of the *payoff fields* be ***valuation fields.*** Further, we suggest ***negotiation fields*** be the *magnetic fields*. We suggest this since there are always two distinct ***negotiation strategies*** for which the payoff represents the payout. Each pair of strategies defines a ***negotiation plane***. Each payout represents a flow of ***negotiation flux*** lines through this *negotiation plane*. In general, since the *negotiation field* is an antisymmetric matrix, at each point we can always find a frame of reference in which the matrix is block diagonal reflecting a disjoint union of *negotiation planes*. If we think of the *negotiation flux* flowing along the ***negotiation plane normal***, we have a generalization of *magnetic field* in any dimension.

We make independent two concepts *game theory* makes dependent as part of equilibrium solutions. Though independent, these two concepts are tied together dynamically. First, the time dependence of the *negotiation flux* through a two-dimensional area is the source of the *valuation field*, Eq. (1.39). Second, we show below that the connection works the other way: the time dependence of the *valuation flux* through the dual $N-1$ dimensional surface is the source of the *negotiation field*, Eq. (1.45).

This set of equations is determined from Eq. (1.43) if one of the coordinates of the "volume" is along the time direction. As before, it is helpful to start with the differential form of the equations:

$$\begin{aligned} \partial_k F^{jk} + \partial_0 F^{j0} &= \kappa j^j \\ \partial_k F^{jk} &= \kappa j^j + \partial_0 F^{0j} \end{aligned} \tag{1.44}$$

We expect two terms from Eq. (1.43). The first is an $N-2$ dimensional surface times the interval Δt along which we integrate the magnetic components F^{mn}. The second is an $N-1$ dimensional surface times the incremental difference (in time) of the electric field along the normal to the surface. This is the analog of ***Ampere's law***:

$$\oint_{\partial K} \tfrac{1}{2!}\frac{F^{mn}}{\kappa}\mathbf{S_{mn}} = \int_K j^m \mathbf{S_m} + \frac{\partial}{\partial t}\int_K \frac{F^{0m}}{\kappa}\mathbf{S_m} \tag{1.45}$$

The right hand side we call the enclosed ***utility current***, including the contribution of the *displacement current* that is induced from *valuation fields* that change in time, flowing **through** the strategic surface.

The boundary can be thought of as the dual of the 2-form created from the wedge product of the surface 1-form and a second intersecting surface 1-form that defines the boundary. The dual of this 2-form is $\mathbf{S_{mn}} = \frac{1}{(N-2)!}\varepsilon_{0p\cdots qmn} dx^p \wedge \cdots \wedge dx^q$, which is a $(N-2)$-form and in general is not a 1-form (except in three dimensions). It does generalize a line integral: the 2-form is the wedge product of two 1-forms and thus represents a subspace normal to the two 1-forms. Its dual is thus an integral element in that subspace.

The left hand side is thus the ***circulation*** of the scaled *negotiation field* F^{mn} **along** the $N-2$ ***strategic boundary*** $\mathbf{S_{mn}}$ that surrounds the *strategic surface*. What passes through the 2-surface circulates in the dual $(N-2)$-surface. The *negotiation field* is between the direction m that is normal to the surface and the direction n that is normal to the intersecting surface defining the boundary. This law states that the *utility currents* that flow **through** the surface generate, or are the source for, the *negotiation fields* that flow **along** the boundary. Each term has units of $[U/T]$.

We thus have a complete set of generalized Maxwell's equations in $N+1$ dimensional space. It is relevant to add that these Maxwell's equations require the current to be conserved, $d*\mathbf{j}=0$, which follows from Eq. (1.42) and (1.35). It is equivalent to $\partial_a j^a = 0$ in differential form and to the more intuitive integral form:

$$\int_{\partial K'} *\mathbf{j} = \int_{K'} d*\mathbf{j} = 0 \Rightarrow \oint_{\partial K} j^m \mathbf{S_m} = -\frac{\partial Q}{\partial t} \tag{1.46}$$

This states that the rate at which the enclosed charge (utility) changes in time is equal to the net current leaving the N dimensional strategic (space) volume K through the enclosing $N-1$ strategic surface ∂K. Charge is not created or destroyed, but flows like a fluid through space.

In this section we have anticipated our adoption of this generalization of the electromagnetic field as the payoff field. We have provided the units that make sense for this generalization. Implicit in our approach is the notion that decision strategies change continuously over time. In the next section we introduce the basis for this anticipation.

1.7 *Continuous change*

In many approaches to decisions (Cf. section 7.3), the key aspect is the uncertainty of the decision and its independence from past events. The approach is to use Bayesian or inverse probability to estimate future behaviors. Yet in modern businesses, the opposite holds: businesses practice continuous process improvement as the basis for improving profits and quality of their products. Improving past behaviors is the key to future successes. We believe as well that ***continuous change*** is a necessary attribute of decisions and can be applied to mixtures of strategies viewed as frequencies (section 1.1). Decision processes can be put into a normal form, albeit ***locally***. For sufficiently short periods of time and sufficiently small strategic variations, the foundational aspects of *game theory* provide a framework for our ***decision process theory***.

To discuss strategic change, we distinguish between changes in choices that occur at a constant rate and those that don't. The latter changes are characterized by acceleration, which is the rate at which strategic rate changes occur. In many cases we are unaware of such *accelerations*. We will refer often to the physical analogy of our perception that as we sit comfortably at home at rest, we are unaware that we are spinning around the North-South axis once every 24 hours and rocketing around the sun every 365 ¼ days. Yet those of us that pay attention to weather are aware of the induced *acceleration* effects attributed to Coriolis, because they contribute to the formation of cyclonic weather conditions (Crawford, 1978). A discussion of dynamic changes involves not only the ability to order our preferences, but the ability to measure the magnitude of change of those preferences and whether the changes occur at a constant or *accelerating* rate. As with weather, we must learn to identify when such acceleration effects are present.

To address these issues we apply what we feel are common sense notions of space (applied to strategic choices) and time to the domain of decision making. We assume that choices that are near to each other strategically or near to each other in time, lead to actions that are correspondingly near. We require a ***measure*** or ***metric*** between decision events: we need the ability to determine what it means for choices to be near or far apart. We need a notion of continuity. We must be able to assign the measure to the rate of changes and thereby distinguish constant from acceleration effects. We then have a basis for our dynamic theory.

Be warned however that in general, common sense notions are not automatically common to all or even mutually consistent. A famous and relevant example is revealed by Einstein's analysis of the Newtonian notions of space and time. Somewhat simplified, Newton's view was that gravity effects propagate instantaneously to all parts of the universe; Einstein, however, maintained that gravity propagates its effects as a signal that goes at the speed of light. Because in a vacuum, the speed of light is very large, the two views are ordinarily indistinguishable. In today's world however, we produce devices and look at phenomena where the effects are easily distinguishable and we must adopt Einstein's view that has yet to be common. This highlights the value of supplementing our common sense with the requirement of a self-consistent framework within which we can formulate our ideas and models.

Einstein's view is relevant to us because it provides a self-consistent and common sense view of *acceleration*. With this view, we show that two notions that we introduce below, ***torsion free*** and ***measure independence***, lead to a determination of the *acceleration effects*. Acceleration is proportional to the forces that determine the dynamics and is thus an important attribute of dynamics in any dynamic theory.

Our starting point for ***decision process theory*** is that decisions are events that generate action and they occur at a moment in *time* and a point in *space*. The foundation of our framework is both topological and geometric. We need the concept of order to indicate preferences and *measure* to determine nearness. We articulate our ideas using relevant information on topology and differential geometry taken from the literature as noted below. We require physical space time to be locally isomorphic to *flat* space, which corresponds to our experience, that curvatures are not ordinarily noticed. This physical assumption presupposes that at each point there is an ***inertial frame of reference***, *i.e.* a frame in which there is no perceived acceleration. This generalizes the common perception that locally the world around us is flat and not moving. This assumes there is an independent set of strategic directions that represents the set of possible choices.

Globally, time and space need not be flat. To explore this we start with the observation that time and space are fundamentally connected by Einstein's notion of cause and effect. A ***cause*** generates a ***signal*** that propagates outwardly from that event and this signal influences future ***effects***. Though we consider a space of *applied* dimensions, we maintain this property that signals propagate with a finite velocity. We choose units (section 1.6) so that the fundamental speed of propagation is unity; distances are measured in units of time. This is analogous to astronomers measuring distances in light-years, the distance light travels in one year. These signals are physically as real as the events that produce them. As such they carry energy and have momentum. They are extended structures in space and time and exhibit stresses analogous to physical media such as fluids.

Since we assume that space is locally *flat*, at every point there is an inertial frame of reference in which a signal generated will propagate outward as a spherical wave. In this frame of reference after any small interval of time dt, the wave resides on a sphere around that point with strategic coordinates dx^a, where a runs through the possible strategic coordinate directions. Since the wave is a spherical wave, the relation between time and space on the sphere is:

$$ds^2 = dt^2 - \sum_a dx^a dx^a = 0 \tag{1.47}$$

We will have occasion throughout this book to indicate sums as we have indicated above that include using the mathematical "sigma" sign for summation and will involve time components. We find this notation cumbersome. With a little practice the following shorthand for quadratic forms is easier to read and understand:

$$ds^2 = g_{ab} dx^a dx^b \tag{1.48}$$

There are three things to keep in mind: first, if there are repeated indices with one upper and the other lower, add a summation "sigma" for that index. Second, allow the indices $\{a, \quad b\}$ to represent not only the allowed strategies but also time. Third, specify all the *tensor* quantities so that the desired expression is replicated. In this case we replicate Eq. (1.47) if the non-zero components are $g_{tt} = 1$ and the strategy components are diagonal and $g_{aa} = -1$. The geometry of this space is not Euclidean but the curved Minkowski space. For our purposes in this book we call spaces with a ***Minkowski metric*** *flat*.

Thus a strategic choice at some instance of time forms a *point* in an $n+1$ dimensional space. We write the space-time point x^a as the $n+1$ *-tuple* $x^a = \{x^0 \quad x^1 \quad \cdots \quad x^n\}$, where the time component is x^0. The components x^a are scalar functions and provide a ***holonomic coordinate basis*** to describe the space-time point. By holonomic, we mean there are scalar functions x^a associated with each coordinate. Surfaces on which each of these scalar functions is constant provide a coordinate basis with which to describe positions at each point in time. The ordinary rules of calculus can be used where the order of partial derivatives yields the same result: $\partial_a \partial_b = \partial_b \partial_a$. The consequence is that changes in scalar quantities between two neighboring points can be computed by traversing along first one coordinate surface a differential direction and then along the second. It does not matter which surface is chosen as first. Not every basis is a coordinate basis. We will see shortly that for more general bases, the order *does* matter.

The existence of a coordinate basis is an important part of our theoretical framework. A familiar example occurs when describing the points on a sphere. One can carry around an actual sphere or map a sphere onto a piece of paper, a two-dimensional projection or chart. For a sufficiently small section of the sphere, the chart provides an accurate flat representation of the sphere where latitude and longitude might be chosen as the coordinate variables. In the general case however, the constant value of the scalar field x^a describes a hyper-surface on which all points have the same coordinate value. In the special case of the sphere this hyper-surface is a circle on the sphere. The example of the sphere demonstrates that the property of being locally flat does not mean the space is globally flat. When the space is not flat, you need more than one chart to navigate the surface.

A new frame of reference is defined by transforming to a set of $n+1$ functions $y^a\left(x^0, x^1, \cdots, x^n\right)$ that depend on the old coordinates. The new frame of reference also provides a holonomic coordinate basis. We require the common sense notion that the specification of pure strategies may not be unique: many equivalent choices are possible. We create a ***decision process theory*** in which any frame of reference is an acceptable choice. We assume ***covariance,*** that the fundamental dynamics will be expressed by a mathematical form that does not depend on which choice we make.

A natural consequence of our dynamic theory is the idea that the strategies change continuously over time (due to a principle of least action). The study of such changes is closely related to how quantities in our *decision process theory* transform: every continuous change can be represented as a transformation. Thus dynamics suggests we investigate the most general frame transformations that might occur. We start by considering the spherical propagation signal. The outgoing signal can be viewed in many other frames. If we relax the condition that we are in a locally flat frame, then we maintain the compact form Eq. (1.48) with $ds = 0$, but the values of the tensor g_{ab} become arbitrary. We say that the "distance" ds is a ***scalar field.*** Since the changes of the tensor g_{ab} may vary from one point to the next, we call such a tensor a ***tensor field*** to emphasize the space time dependency. We must be able to compute the values of this tensor in an arbitrary frame. We see that the tensor describes the cause and effect relationships since it prescribes the behavior of the outgoing signal. We therefore describe in more detail the transformation properties between frames and how they lead to an understanding of the cause-effect signals and how this relates to the tensor g_{ab}.

We recall the rules from calculus that differential changes of variables, which are important for specifying cause and effect in Eq. (1.48), can be determined from the partial derivatives and the chain rule:

$$dy^a = \frac{\partial y^a\left(x^0, x^1, \cdots, x^n\right)}{\partial x^b} dx^b = \partial_b y^a dx^b \tag{1.49}$$

Note our use of the summation convention to simplify the writing and our introduction of a more compact notation to indicate the partial derivatives. Though the expression can be a laborious computation, the essential conceptual point is that the new differentials are linearly related to the old differentials.

We expand the concept of a frame transformation by allowing *any* linear transformation of the differentials, including those that are non-holonomic. The resultant expression $\mathbf{E}$ we call a 1-form using the name from the theory of differential geometry (section 1.6), which, when these infinitesimal quantities are summed over a path, provides the basis for defining a ***line integral***:

$$\mathbf{E}^\mathbf{a} = E^a{}_b dx^b \tag{1.50}$$

This expression is a ***general coordinate basis*** and represents the most general linear frame transformation at a point that might occur in our dynamic theory of decisions.

We note that for the special case that the 1-form consists of coordinates that are perfect differentials, we have $\partial_c E^a{}_b = \partial_b E^a{}_c$, which follows from the equality of the mixed derivatives in Eq. (1.49), $\partial_c \partial_b y^a = \partial_b \partial_c y^a$. A non-holonomic frame is one in which $\partial_c E^a{}_b = \partial_b E^a{}_c$ for every pair of coordinate values is not true for at least one pair. The usual rules of calculus don't hold. Changes to fields now depend on the path taken as a consequence of describing effects in these frames.

We assumed that at each point there is a frame in which the cause and effect signal propagates outwardly as a spherical wave. In a general frame $\mathbf{E}^\mathbf{a}$ we assert that this implies that the propagation is determined by the vanishing of the quadratic form $ds^2 = g_{ab}\mathbf{E}^\mathbf{a}\mathbf{E}^\mathbf{b}$ constructed from the frames. The coefficients are determined by the transformation.

Cause and effect must apply to all physical phenomena. We expect that regularities of physical phenomena will involve the rates of change of variables, such as a ***scalar field*** x^a, a ***vector field*** X^α, or an order 2 ***tensor field*** g_{ab}. There will be higher order tensor fields to consider as well. We need rules that apply to any tensor field and its rates of change. In the frame in which space is locally flat, rates of change are determined by ordinary calculus, by the derivatives. Transformation to ***rotating frames*** requires that the effects of the acceleration be included in the definition of the rates of change. We again recall that the Coriolis force due to living on a spinning sphere (earth) is an example of a *rotating frame* effect. These considerations thus are really common sense additions that must be part of the *decision process theory*.

The basic rules to include *rotating frame* effects for any tensor field can be generalized based on how we treat the rate of change of a vector field. The rate of change of a vector field along a path parameterized by a scalar τ is the difference of the field at two different points divided by the differential change $d\tau$ along the path. In dealing with *rotating frames* we must be concerned with Coriolis-type forces: we must be clear about how we compare vectors at two different points. In physical applications, we make the comparison by ***parallel translating*** the initial vector along the curve to the comparison point. This removes the effect of the *rotating frame*, exposing only the actual change in the vector field. As a result, the rate of change of the vector field will be a differential change of its values plus a contribution due to the *frame rotation*, which linearly mixes the vector components:

$$DX^a = dX^a + \boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}} X^b \tag{1.51}$$

This defines the ***covariant derivative***. The linear mix or ***orientation potential*** $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}}$ is a 1-form (remember that the 1-form is determined by the frame properties $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}} = \omega^a{}_{bc} d\mathbf{x}^{\mathbf{c}}$). The differential change is determined by the ordinary partial derivative 1-form $dX^a = \partial_b X^a dx^b$. A useful variant of this form is to define the covariant generalization of the partial derivative as $DX^a = X^a{}_{;b} dx^b$:

$$X^a{}_{;b} = \partial_b X^a + \omega^a{}_{cb} X^c \tag{1.52}$$

Rules that are expressed in terms of the covariant derivative will be ***covariant*** in that they will have the same form in any frame of reference. This is an important and required aspect of any physical theoretical foundation including ours.

We transform from a holonomic coordinate basis to a general coordinate basis using Eq. (1.50). In the general coordinate basis, the covariant derivative has the same form Eq. (1.51). Therefore the orientation potential 1-form $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}}$ will now be expressed in terms of the new coordinates with $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}} = \omega^a{}_{bc} \mathbf{E}^{\mathbf{c}}$. The partial derivative ∂_a transforms to the ***frame derivative*** $\Delta_a = E_a{}^b \partial_b$, which depends on the inverse $E^c{}_b$ of the frame transformation Eq. (1.50):

$$dX^a = \Delta_b X^a \mathbf{E}^{\mathbf{b}} \tag{1.53}$$

The second term of Eq. (1.51) is zero if globally there is no frame rotation. In the general case, the properties of this 1-form are completely determined by the two following assumptions.

First, for any general coordinate basis components $\mathbf{E}^{\mathbf{a}}$, the orientation of each frame is unchanged under ***coordinate displacement***. We express this assumption compactly as saying frames are ***torsion free***:

$$D\mathbf{E}^{\mathbf{a}} = d\mathbf{E}^{\mathbf{a}} + \boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}} \wedge \mathbf{E}^{\mathbf{b}} = 0 \tag{1.54}$$

In a space that is ***locally flat,*** all frames will be *torsion free*. The 1-form has properties of a vector field so we expect the two terms for the differential of a vector 1-form. The 1-form represents a differentially short path segment of a curve in space. The line integrals are constructed from the integration of such segments along a curve. The differential represents the *coordinate displacement* of the entire segment and so spans an area. In three-dimensional vector calculus, the area is the cross product of two differential vectors, one along the segment and the other along the displacement path.

In higher dimensions the area is determined by the antisymmetric ***wedge product*** of two 1-forms, called a 2-form, defined in terms of the measure for area:

$$\mathbf{A} \wedge \mathbf{B} = A_a dx^a \wedge B_b dx^b = A_a B_b dx^a \wedge dx^b = \tfrac{1}{2}\left(A_a B_b - B_a A_b\right) dx^a dx^b \tag{1.55}$$

The study of forms is therefore the study of measures of length, area, volume, etc. Based on these forms, the first term of Eq. (1.54) is determined by the ***structure constants*** $C^a{}_{bc} = -C^a{}_{cb}$ defined in terms of the partial derivatives of the frame transformation components:

$$d\mathbf{E}^{\mathbf{a}} = \tfrac{1}{2}\left(\partial_b E^a{}_c - \partial_c E^a{}_b\right) dx^b \wedge dx^c \equiv \tfrac{1}{2} C^a{}_{bc} \mathbf{E}^{\mathbf{b}} \wedge \mathbf{E}^{\mathbf{c}} \tag{1.56}$$

The second term of Eq. (1.54) states that *coordinate displacement* of a short segment has an additional contribution for a rotating frame.

Any general frame under *coordinate displacement* is unchanged as expressed by Eq. (1.54). For rotating frames this provides one relationship between the linear transformation $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}}$ 1-forms and the structure constants $C^a{}_{bc}$. We thus also call a general coordinate basis a ***fixed coordinate basis***.

Second, the key assumption is ***measure independence***: the covariant derivative of Eq. (1.48) is zero. An outgoing spherical signal wave must be a frame independent statement. This implies that the covariant derivative of the tensor g_{ab} is zero in any frame. The covariant derivative of this tensor transforms in a dual manner to the symmetric product of two vectors. Based on this consideration, the covariant derivative can be shown to be:

$$Dg_{ab} = dg_{ab} - \boldsymbol{\omega}^{\mathbf{c}}{}_{\mathbf{a}} g_{cb} - \boldsymbol{\omega}^{\mathbf{c}}{}_{\mathbf{b}} g_{ac} \tag{1.57}$$

We require $Dg_{ab} = 0$.

These considerations lead to an important result that provides a relationship between the ordinary (frame) derivatives of the tensor g_{ab} and the orientation potential 1-form $\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$, which when combined with the first assumption provides the explicit relationship of *orientation potential* components and the *structure constant* coefficients $C^a{}_{bc}$ in Eq. (1.56):

$$\omega_{abc} = \tfrac{1}{2}\left(C_{abc} + C_{bca} + C_{cba}\right) + \tfrac{1}{2}\left(-\Delta_a g_{bc} + \Delta_b g_{ca} + \Delta_c g_{ab}\right) \tag{1.58}$$

It is instructive to consider the orientation potentials for a ***coordinate basis***:

$$\omega_{abc} = \tfrac{1}{2}\left(-\partial_a g_{bc} + \partial_b g_{ca} + \partial_c g_{ab}\right) \tag{1.59}$$

It simplifies because the differential of the coordinate frame is zero $d\left(dx^a\right) = 0$, which implies that the structure coefficients $C_{abc} = 0$ are also zero. The orientation potential in this basis, also called the ***connection***, is symmetric in the last two indices, $\omega_{acb} = \omega_{abc}$.

Another special case is the ***orthonormal coordinate basis*** in which by definition, the metric is everywhere equal to the Minkowski metric and so the frame derivatives are zero:

$$\omega_{abc} = \tfrac{1}{2}\left(C_{abc} + C_{bca} + C_{cba}\right) \tag{1.60}$$

The orientation potential in this basis is antisymmetric in the first two indices $\omega_{bac} = -\omega_{abc}$, which follows from the antisymmetry of the structure constants.

1.8 *Geometric structures*

We have stated that in *decision process theory,* the rules for decisions should be ***covariant***. The basis for this comes from utility theory in *game theory*, in which preferences are determined only up to linear transformations (section 7.4). Thus for any given event, we express the theory in such a way that the rules are independent of linear transformations. Theories that are invariant under transformations have characteristic geometric structures, such as scalars, vectors and tensors, which we will define in more detail below. We start the discussion by laying out in detail what this covariance implies.

We identify a ***geometric structure*** as a set of related functions in the theory that maintain their form under transformation. The simplest example is a single function that transforms into itself. We define a field ϕ to be a ***scalar field*** if $\phi' = \phi$, which indicates no change under the transformation other than expressing the field in terms of new variables.

The next more complicated structure is a collection of functions in 1-1 correspondence to the number of space time dimensions. The simplest example is that of functions that define a frame. For each direction b, the frame transformation components $E^a{}_b$ form the components of a *contravariant vector*[3] in the new basis: the collection of such vectors is $\mathbf{E^a}$. (One can equally well look at each direction a, and the frame transformation is a set of *covariant vectors*[4] in the old basis; similar statements can be made about the inverse frame transformation $E_a{}^b$.) We consider a frame transformation $\boldsymbol{\lambda}\mathbf{E}$ a product between the matrix $\boldsymbol{\lambda}$, which is a frame transformation in its own right, and the frame components $\mathbf{E}$ that transforms a set of frames into a new set (denoted by the prime):

$$\mathbf{E'^a} = \lambda^a{}_b \mathbf{E^b} \tag{1.61}$$

As with the scalar field, the new frame components are also expressed in terms of the transformed coordinate values.

We generalize this and define X^a to be a ***vector field*** if the components transform to the matrix product $\mathbf{X'} = \boldsymbol{\lambda}\mathbf{X}$, and are expressed in the new variables. We will have occasion to distinguish between two types of vector fields. Following the literature we call this one a ***contravariant vector***. The 1-form

[3] Defined in the next paragraph.
[4] Defined in the next paragraph.

$\boldsymbol{\omega} = \omega_a E^a$ has components ω_a that are dual to contravariant vectors in that they transform according to the inverse matrix $\boldsymbol{\lambda}^{-1}$. These are components of a ***covariant vector***.

A ***tensor field*** is the generic name for *geometric structures* that include *scalar fields*, *vector fields* and more general fields in which there is some number of contravariant components and some number of covariant components that separately transform as the corresponding contravariant or covariant vectors respectively. We can now be more precise about our requirement that *decision process theory* be covariant. We consider only laws that are expressed in terms of such *geometric structures* or *tensors*.

In any theory that allows frame rotations, it will not be generally true that geometric structures at one point carry over to the same geometric structure at some other point. Yet that is the type of theory we require. The general class of such theories is called a gauge theory by for example (Hawking & Ellis, 1973) and a fibre bundle by mathematicians, (Steenrod, 1951). We require the following notions.

We require that the transformation matrix is a continuous function of position and we refer to it as a ***gauge transformation***. Our ***decision process theory*** is a member of the class of gauge theories that have been well-studied. The orientation potential 1-forms $\boldsymbol{\omega}^{\mathbf{a}}{}_{\mathbf{b}}$ form a matrix $\boldsymbol{\omega}$ of 1-forms. Though these potentials are not components of a tensor, they do have a transformation rule that makes sense:

$$\boldsymbol{\omega}'(C)\boldsymbol{\lambda}(x) = \boldsymbol{\lambda}(y)\boldsymbol{\omega}(C) \tag{1.62}$$

In words, a transformation to a new frame followed by a ***parallel translation*** along a curve C is the same as a *parallel translation* along the curve C followed by a transformation to the new frame. This provides the connection between the geometric structure at one point and at some other point. We show this in more detail.

When the *parallel translation* is differentially small, from $y - dy$ to dy, then this transformation rule, which also holds for each frame transformation, is:

$$\boldsymbol{\omega}'(y) = \boldsymbol{\lambda}(y)\boldsymbol{\omega}(y)\boldsymbol{\lambda}^{-1}(y) + \boldsymbol{\lambda}(y)d\boldsymbol{\lambda}^{-1}(y) \tag{1.63}$$

The first term is the transformation of a mixed tensor. The second term indicates that for *rotating frames*, the orientation potential is not a mixed tensor.

It is also true that the ordinary derivative of a *contravariant vector field* is not a tensor since the transformation of the differential of a *contravariant vector field* is:

$$d\mathbf{X}' = \boldsymbol{\lambda} d\mathbf{X} + d\boldsymbol{\lambda}\boldsymbol{\lambda}^{-1}\mathbf{X}' \tag{1.64}$$

Putting this together with Eq. (1.63), we see that the covariant derivative Eq. (1.51), formed from the *parallel translation* of the *contravariant vector field*, does transform like a vector field. The transformation components of Eq. (1.63) that don't transform as a vector field cancel the corresponding components of Eq. (1.51) of the differential of the vector field:

$$D\mathbf{X}' = \boldsymbol{\lambda} d\mathbf{X} + d\boldsymbol{\lambda}\boldsymbol{\lambda}^{-1}\mathbf{X}' + \boldsymbol{\omega}'\mathbf{X}' = \boldsymbol{\lambda} d\mathbf{X} + \boldsymbol{\lambda}\boldsymbol{\omega}\mathbf{X} = \boldsymbol{\lambda} D\mathbf{X} \tag{1.65}$$

In particular this holds for the *torsion free* coordinate basis vectors:

$$D\mathbf{E}' = \boldsymbol{\lambda} D\mathbf{E} \tag{1.66}$$

Frames are geometric structures defined over the whole space; they transform as vector fields.

We thus have a proof that local gauge invariance for a space that is locally flat (Minkowski) implies that every frame is *torsion free*. This provides a justification for Eq. (1.54). Because the derivative D that results from *parallel translation* preserves the tensor field transformation, the name ***covariant derivative*** is appropriate. It is a differential operator that preserves the *geometric structures* of the theory.

Although the orientation potentials are not tensors or geometric structures that are preserved, they are associated with such structures and are physically real fields. Preserved structures are by definition, gauge invariant. It is proved in the literature that the flux that goes through a closed line integral formed from the *orientation potential* 1-form matrix is equal to a gauge invariant 2-form:

$$\mathfrak{R} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} \tag{1.67}$$

This ***orientation flux field*** carries energy and momentum through the 2-surface, is a tensor field and provides the physical field that propagates the cause to the effect. In physical theories it is called the ***curvature tensor***. If we can relate this physical *orientation flux field* to the forces, we will have specified *decision process theory* as a dynamical theory. We make a start at this in the next section where we explain how *hidden dimensions* lead to the payoff fields suggested in Eq. (1.26) as *frame rotation effects*.

1.9 ***Hidden dimensions***

The challenge is bridging the view we have been developing that approaches decisions in terms of the strategies or choices available to a player, considered as *applied* dimensions of a many body system, and the dynamic content suggested from *game theory* considering payoffs for each player. We find similarities with a challenge posed in physics to unify the applied forces viewed in space and time with internal symmetries such as electromagnetic fields. This is by no means obvious. Suffice to say at this introductory point that a solution by (Kaluza, 1921) and (Klein, 1956) was proposed almost a century ago to consider the electromagnetic fields as an *internal symmetry* or *hidden symmetry* represented by an additional dimension to be added to ordinary space and time. This only works if the additional dimension plays no active role in the dynamics in a sense that we make precise below.

We shall see that the analogous mechanism for *decision process theory* is to add a player-specific (internal) strategy for each player, which we call ***inactive*** from the usual perspective of strategies. The usual strategies we shall call ***active***. This approach we believe resolves the conceptual problems and provides the necessary bridge. We use this as the framework for our ***decision process theory***.

It is thus our view that decisions are carried out jointly by one or more subjects who act based on their personal knowledge of past actions. There is some attribute or behavior of each player or agent that is ***persistent***; that does not change with time. The notion of persistence is tied with our notion above that the player-specific strategy is inactive. In contrast to previous authors, we view that the players' active behaviors or strategic choices need not be persistent, rational or necessarily converge to some equilibrium flow. Their choices in general are ***variable*** precisely because they learn from past outcomes.

Following this idea, we add inactive dimensions that we associate with the players in the decision, to the active strategy dimensions associated with the pure strategies. In a coordinate basis, the most general form of the ***line element*** Eq. (1.48) with hidden dimensions ξ^j is:

$$ds^2 = \gamma_{jk}\left(d\xi^j + A_a^j dx^a\right)\left(d\xi^k + A_b^k dx^b\right) + g_{ab}dx^a dx^b \qquad (1.68)$$

We indicate active strategies by the indices $\{a,b,\cdots\}$ and the inactive or hidden strategies by the indices $\{j,k,\cdots\}$. We define ***hidden*** to mean that the ***inactive metric components*** γ_{jk}, ***payoff potentials*** A_a^j and ***active metric components*** g_{ab}, are independent of the ***inactive dimensions*** ξ^j. For the purpose of this discussion time is an active strategy. The *hidden* or *inactive dimensions* play a restricted dynamic role.

In differential geometry, when there is a frame in which the metric elements have the form Eq. (1.68) and are independent of the dimension ξ^j, then there is an ***isometry*** associated with that dimension: the metric does not depend on that dimension. As part of our *decision process theory* we postulate that there is at least one isometry for each player in a decision. To the extent that players may exhibit schizophrenic tendencies, there may be more than one isometry for each player.

We view players as being independent and so the isometries lead to metric elements that are simultaneously independent of all inactive dimensions. For the mathematically oriented reader, each isometry implies the existence of a vector field. These vector fields when considered as operators form a Lie Algebra. The associated group for these players is a commuting group: two successive transformations yield the same result independent of order.

The existence of these *isometries* leads naturally to the notion of payoff fields, which we now illustrate. We observe that in the expression for the line element Eq. (1.68), in what we call the ***normal-form coordinate basis***, the natural choice for a basis is a coordinate or ***exact*** basis for the active directions, $\mathbf{U^a} = dx^a$ by which we mean $d\mathbf{U^a} = 0$, but not for the inactive dimensions $\mathbf{U^j} = d\xi^j + A_a^j dx^a$, by which we mean $d\mathbf{U^j} \neq 0$:

$$\begin{aligned} \mathbf{U}^a &= dx^a \\ \mathbf{U}^j &= d\xi^j + A_a^j dx^a \end{aligned} \tag{1.69}$$

We show that the payoff potentials $A^j{}_a$ lead to payoff fields $F^j{}_{ab}$, Eq. (1.73) below.

We first note that these frames remain unchanged if to the inactive dimension we add an arbitrary function Λ^j of the active variables $\xi^j \to \xi^j + \Lambda^j(x)$ and to the payoff potential subtract a gradient of this function, $A_a^j \to A_a^j - \partial_a \Lambda^j$. Such a transformation is a ***gauge transformation associated with player*** j. It is a subset of the general class of gauge transformations Eq. (1.62). As such we expect laws to be expressed in terms of tensors that are invariant under such a transformation. By construction the inactive and active metric elements are independent of these transformations.

To identify the payoff potential with the payoff matrix that occurs in the flow equation Eq. (1.13) suggested by game theory we compute the behavior of ***geodesic curves*** defined in terms of the (vector) flows:

$$\begin{aligned} V^a &= \frac{dx^a}{d\tau} \\ V^j &= \frac{d\xi^j}{d\tau} \\ \frac{DV^a}{\partial\tau} &\equiv V^a{}_{;b} V^b + V^a{}_{;k} V^k = 0 \\ \frac{DV^j}{\partial\tau} &\equiv V^j{}_{;a} V^a + V^j{}_{;k} V^k = 0 \end{aligned} \tag{1.70}$$

A geodesic curve is characterized in geometric terms as a curve that is the shortest distance between two points. For example on the surface of the sphere, the geodesics are great-circles. Though simple in form, in the *normal-form coordinate basis* there will be *orientation potentials* in addition to the derivatives of the flow.

In order to relate these *orientation potentials* to the metric components in Eq. (1.68) we use the concepts described in the previous section. We use the inverse of the transformations in Eq. (1.69) as well as the frame partial derivatives:

$$\begin{aligned} \mathbf{U} &= \left(U_\mu^\alpha\right) = \begin{pmatrix} U_k^j & U_b^j \\ U_k^a & U_b^a \end{pmatrix} = \begin{pmatrix} \delta_k^j & A_b^j \\ 0 & \delta_b^a \end{pmatrix} \\ \mathbf{U}^{-1} &= \left(U_\alpha^\mu\right) = \begin{pmatrix} U_j^k & U_b^k \\ U_j^a & U_b^a \end{pmatrix} = \begin{pmatrix} \delta_j^k & -A_b^k \\ 0 & \delta_b^a \end{pmatrix} \\ \Delta_j &= U_j^\mu \partial_\mu = \partial_j \\ \Delta_a &= U_a^\mu \partial_\mu = \partial_a - A_a^k \partial_k \end{aligned} \tag{1.71}$$

On the space of functions that are independent of the inactive coordinates, we see that Δ_j gives zero and $\Delta_a = \partial_a$. The line element Eq. (1.68) can be written in the normal-form coordinate basis:

$$ds^2 = \gamma^{jk} \mathbf{U}_j \mathbf{U}_k + g_{ab} \mathbf{U}^a \mathbf{U}^b = \gamma_{jk} \mathbf{U}^j \mathbf{U}^k + g_{ab} \mathbf{U}^a \mathbf{U}^b \tag{1.72}$$

In this basis, the inactive and active metrics are orthogonal.

We obtain the orientation potentials using Eq. (1.58) by first computing the structure constants[5] $C^\alpha{}_{\beta\gamma}$ from examining the 2-forms constructed from the differentials of the frames, where the matrix $\mathbf{F^j}$ is shorthand for the ***payoff field*** $F^j{}_{ab} = \partial_a A_b^j - \partial_b A_a^j$ defined in terms of the payoff potential:

[5] Greek indices cover both active and inactive dimensions.

$$
\begin{aligned}
&d\mathbf{U}^a = 0 \\
&d\mathbf{U}^j = \mathbf{F}^j = \tfrac{1}{2} C^j{}_{ab} \mathbf{U}^a \wedge \mathbf{U}^b \\
&C^a{}_{\beta\gamma} = 0 \\
&C^j{}_{kl} = C^j{}_{ka} = 0 \\
&C^j{}_{ab} = F^j_{ab}
\end{aligned} \tag{1.73}
$$

In other words the active components are *exact* and the inactive components are orthogonal to the active components, but internally are not orthonormal. So the active orientation potentials are determined by the active metric. We still need to verify that the payoff field that appears here is also the one associated with the flow equation. To do this we need the orientation potentials that appear in Eq. (1.70).

The orientation potentials that have one or more inactive components are determined algebraically from Eq. (1.58):

$$
\begin{aligned}
&\varpi_{\alpha\beta\gamma} = \tfrac{1}{2}\left(C_{\alpha\beta\gamma} + C_{\beta\gamma\alpha} - C_{\gamma\alpha\beta}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_\alpha \gamma_{\beta\gamma} + \mathbf{\Delta}_\beta \gamma_{\gamma\alpha} + \mathbf{\Delta}_\gamma \gamma_{\alpha\beta}\right) \\
&\varpi_{jkl} = 0 \\
&\varpi_{ajk} = \tfrac{1}{2}\left(C_{ajk} + C_{jka} - C_{kaj}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_a \gamma_{jk} + \mathbf{\Delta}_j \gamma_{ka} + \mathbf{\Delta}_k \gamma_{aj}\right) = -\tfrac{1}{2}\partial_a \gamma_{jk} \\
&\varpi_{jak} = \tfrac{1}{2}\left(C_{jak} + C_{akj} - C_{kja}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_j \gamma_{ak} + \mathbf{\Delta}_a \gamma_{kj} + \mathbf{\Delta}_k \gamma_{ja}\right) = \tfrac{1}{2}\partial_a \gamma_{kj} \\
&\varpi_{jka} = \tfrac{1}{2}\left(C_{jka} + C_{kaj} - C_{ajk}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_j \gamma_{ka} + \mathbf{\Delta}_k \gamma_{aj} + \mathbf{\Delta}_a \gamma_{jk}\right) = \tfrac{1}{2}\partial_a \gamma_{jk} \\
&\varpi_{jab} = \tfrac{1}{2}\left(C_{jab} + C_{abj} - C_{bja}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_j \gamma_{ab} + \mathbf{\Delta}_a \gamma_{bj} + \mathbf{\Delta}_b \gamma_{ja}\right) = \tfrac{1}{2}\gamma_{jk} F^k_{ab} \\
&\varpi_{ajb} = \tfrac{1}{2}\left(C_{ajb} + C_{jba} - C_{baj}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_a \gamma_{jb} + \mathbf{\Delta}_j \gamma_{ba} + \mathbf{\Delta}_b \gamma_{aj}\right) = -\tfrac{1}{2}\gamma_{jk} F^k_{ab} \\
&\varpi_{abj} = \tfrac{1}{2}\left(C_{abj} + C_{bja} - C_{jab}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_a \gamma_{bj} + \mathbf{\Delta}_b \gamma_{ja} + \mathbf{\Delta}_j \gamma_{ab}\right) = -\tfrac{1}{2}\gamma_{jk} F^k_{ab} \\
&\varpi_{abc} = \tfrac{1}{2}\left(-\partial_a g_{bc} + \partial_b g_{ca} + \partial_c g_{ab}\right)
\end{aligned} \tag{1.74}
$$

This result is insightful. The *orientation potentials* with three inactive coordinates are zero. The orientation potentials with three active dimensions depend entirely on the active metric as if the inactive metric and payoff potentials were absent. The remaining orientation potentials are determined either by the gradient of the inactive metric or the *payoff field* for each player.

The *payoff fields* generate ***vorticity***. We use the components of the orientation potential in the expression for the geodesics Eq. (1.52) for the active flow:

$$
\begin{aligned}
&\frac{DV^a}{\partial\tau} = V^a{}_{;b}\frac{dx^b}{d\tau} + V^a{}_{;j}\frac{d\xi^j}{d\tau} = V^a{}_{;b}V^b + V^a{}_{;j}V^j = 0 \\
&\frac{dV^a}{d\tau} + \varpi^a{}_{cb}V^cV^b + \varpi^a{}_{jb}V^jV^b + \varpi^a{}_{bj}V^bV^j + \varpi^a{}_{kj}V^kV^j = 0 \\
&g_{ab}\frac{dV^b}{d\tau} = -\varpi_{abc}V^bV^c + V_kF^k_{ab}V^b - \tfrac{1}{2}V_jV_k\partial_a\gamma^{jk}
\end{aligned} \tag{1.75}
$$

We thus obtain the needed result that the game theory form Eq. (1.13) is obtained when both active and inactive metric tensors are constants. From Eq. (1.26), we identify the inactive flow V_k roughly with $-e_k/m$, with a different conserved charge for each player. Note that the game theory form is a special case, not part of the theoretical foundation. The theoretical foundation gives us insight into the mechanisms of decision behaviors. In particular, it has been shown (Thomas G. H., 2006) that the decision flow above describes a ***vortex*** behavior. We use our approach to understand better the forces that generate that behavior from individual choices.

We see that the *payoff fields* have two distinct contributions based on the decomposition of the *payoff field* into its *valuation and negotiation fields*. There is one force (acceleration) along the ***valuation field*** direction and is proportional to the *utility density*. There is a second force due to the ***negotiation field***. Unless the flow is in the two-dimensional plane defined by the *negotiation strategies*, this force is zero. The force is *magnetic:* it is orthogonal to the flow and to the normal to the plane. Negotiation leads to change when one of the strategies is along the direction you want to go and the other is an independent alternative with a non-zero payment.

We expect additional contributions both from the orientation potentials in the active space ω_{abc} as well as from the scalar gradients of the inactive metric. We see that the inactive space though hidden contributes to the dynamics, albeit in a restricted way. As an example from the physical domain, we note that the rotation of the earth is also hidden because we think we are at rest; the Coriolis force is the evidence from the orientation potentials that we are spinning.

The inactive flow appears as a "charge" density[6] for player k, which we can call the ***interest flow*** for player k. We now extend game theory by exploring the properties of the *player interest flow*. We demonstrate that its value stays the same (is conserved) along a geodesic. An attribute of a spinning body is the conservation of angular momentum. There is an analogy to decision theory. The geodesics along the hidden directions can be computed from Eq. (1.52):

$$\begin{aligned}
&\frac{DV^j}{\partial\tau} = V^j{}_{;a}V^a + V^j{}_{;k}V^k = \frac{dV^j}{d\tau} + \varpi^j{}_{cb}V^cV^b + \varpi^j{}_{kb}V^kV^b + \varpi^j{}_{ck}V^cV^k = 0 \\
&\frac{dV^j}{d\tau} + \gamma^{jl}\frac{d\gamma_{lk}}{d\tau}V^k = 0 \\
&\frac{dV_k}{d\tau} = 0
\end{aligned} \tag{1.76}$$

In other words the *interest flow* V_j is conserved or ***persistent*** along a geodesic. There are no dynamic changes along the hidden direction since its "momentum" is constant. It is a general rule that for each isometry, in addition to a payoff matrix, there is a conservation law and a ***conserved charge***. In this case we can say that the player's *interest flow* is *persistent*. Based on these equations, we can also say that there will be two associated forces: a Coriolis force and a centripetal force.

1.10 ***Outcomes***

We have completed our first pass in-depth description of *decision process theory*. It is kinematic in content since we have not addressed inertial forces, which will be covered in the next chapter. We have created a *decision process theory* that contains significant aspects of the theory of games as a special case, but the framework is by no means derived from the theory of games. Our goal is to present a comprehensive and robust framework in which to discuss decisions. By their nature decisions involve complex strategies and agents, which we have identified with the game theory name player. These players make choices that in fact may change over time and space. We have been led to a framework in which the strategic choices are active dimensions of space, the players provide one or more inactive strategies each of which is hidden but manifests as a payoff field associated with that player. The geodesics of this geometry provide a behavior that replaces Eq. (1.13), which in game theory represents not dynamics but a mechanistic device to identify the equilibrium strategies.

We believe our *decision process theory* is not only a realistic extension to the discussion of decisions from *game theory*, but a framework that maintains consistency with the laws of the physical world. We reaffirm our belief that decisions are acts in this real world and can be understood using the ideas and language we have learned from a study of that real world. Though the subject matter involves human and society interactions, we believe the empirical method applies. We assert that the best way to study decisions is to have a robust and self-consistent *decision process theory* within which we frame our arguments. We wish to avoid what we see as a common practice of using a piecemeal approach to interesting problems in which one tailors a model to fit the facts.

---

[6] To be more precise, the inactive flow is the ratio of the charge to the mass.

Having come this far however, we believe we must extend our *decision process theory* further to be able to include other interesting aspects of human decision making. There is no question but that there are inertial forces at play in the realm of decision making and that these inertial forces are so far absent in our theory. In the next chapter we explore such inertial forces using as a learning example, the behavior of fluids and suggesting how those effects may be included in the theory of decisions.

We here summarize the chapter from the standpoint of using it in a course. By the end of the chapter, the student should be able to do the following using this framework:

- Identify real-world decision processes
- Put decision processes into normal form
- Identify the persistent (***inactive***) strategies
- Identify the variable (***active***) strategies
- Identify and apply Physical Decision Theory to local behaviors
- Understand global behaviors of the theory
- For specialized cases be able to apply global theory to realistic decision processes
- Be able to identify and use the underlying connections of physical theory to gain insights into decision processes
- Be able to classify general decision processes in terms of a standard taxonomy.

The attainment of these outcomes will be facilitated by doing the exercises in the next section. As a help, the detailed outcomes expected of the student are listed below.

- From section 1.1, as part of the foundational aspects of *game theory*, the student should be able to identify the intensive form of a decision process and isolate the strategic choices each agent has available.
- From section 1.2, the student should be able to recall the essential concepts of Newtonian mechanics.
- From section 1.3 the student should be able to appreciate the reformulation of Lagrange that allows Newtonian mechanics to be effectively applied to the solution of practical problems that involve constraints. Such problems are related to the types of problems we encounter in making decisions: many variables with man constraints in which the essential aspects of the problem are captured by a subset of the variables.
- From section 1.4 the student should gain an appreciation of how it is possible that decision processes as described by game theory might be related to equations that describe physical phenomena.
- From section 1.5 the student sees the mathematical connection between payoff fields in game theory and the electromagnetic fields in physics.
- From section 1.6, Maxwell equations are expressed in integral form and units are proposed for the generalization of these equations to *decision process theory*.
- From section 1.7 the student should learn the essential point that decisions generate actions and are influenced by actions. Furthermore, actions are related by cause and effect, which propagate at a finite speed. The geometric properties of these effects are determined by the common sense properties that transformations are *torsion free* and *measure independent*, and these properties determine the *orientation potential* in terms of the metric and structure constants.
- From section 1.8, the student will have learned that the concept of covariance is that the rules of the theory are expressed in terms of *geometric structures* that maintain their property over time and space. The key such properties are scalar fields, vector fields and tensor fields. A consequence of the theory is that cause and effect are carried by the ***orientation flux field***, called the curvature tensor in differential geometry. It measures the degree to which global space time is not flat.
- From section 1.9, the student will have learned that *hidden dimensions* provide the mechanism for frame orientation effects. Like the Coriolis acceleration, payoffs reflect

accelerations associated with hidden dimensions and provide a *game theory* mechanism for strategic behavior in *decision process theory*. They also provide a persistent conserved quantity analogous to charge conservation in physics.

1.11 *Exercises*

1. A childhood game is TIC-TAC-TOE. This game has two players who make an X or an O on a piece of paper on which there is a 3x3 grid. The players take turns and can make only a single mark. The ***extensive form*** of a game is simply the moves each player makes. For example player 1 picks a mark, which he must stick to in all subsequent moves, puts his mark, say an X, in the 1-1 position. Player 2 now must use the opposite mark, say an O, and picks a spot say in the 1-2 position. Player 1 puts an X in the 1-3 position. The game continues in this fashion. The game ends when either all the spots are filled or when either player has his or her marks in a line consisting of a row, a column or along a diagonal. If all the spots are filled without three marks in a line, the game is tied and is called a cat's game. Write out each possible scenario for the two players.
2. In exercise 1, the extensive form of TIC-TAC-TOE leads to a very large number of possibilities. In looking at the decision tree identify the paths that lead to a win and those that don't. How many paths lead to a win? We call each decision path for a player the pure strategy and the enumeration of the pure strategies of each player the ***intensive form*** of the game. Is there a strategy (strategies) that will ensure a win or cat's game every time?
3. A company has two product lines: it makes razors and it makes blades. Each product line employs a vast number of people and requires substantial resources. Discuss the extent to which the choice between making razors, blades or both is a decision process in extensive form.
4. From Eq. (1.1) show the three Newtonian laws: a body at rest remains at rest, a body in motion remains in motion and for every action there is an equal and opposite reaction.
5. Use Eq. (1.1) and the fundamental inverse square law of gravity to show that the motion of a planet around the Sun is an ellipse.
6. Using the *principle of least action*, determine how long it takes for a body of mass 150 lb. to reach the ground starting from a height 10 yards assuming the body falls freely (ignore air resistance).
7. Using the *principle of least action*, determine how long it takes for a body of mass 150 lb. to reach the ground starting from a height 10 yards assuming the body is constrained to ride a track without resistance whose equation of height $h(d)$ is known as a function of horizontal distance d .
8. The ***tragedy of the commons*** (*Cf.* section 7.2) refers to a time in which land was held in common and those surrounding the land mutually agreed to limited use. If anyone person abused that agreement, few might know or be impacted. However if all those surrounding the land abused the agreement, the land would become overused and useless to all. In what way is the tragedy of the commons related to the prisoner's dilemma.
9. Derive the form of the *orientation potentials*, Eq. (1.74).
10. Derive the active geodesic flow (without sources), Eq. (1.75).
11. Derive the inactive geodesic flow (without sources), Eq. (1.76).
12. The Coriolis acceleration can be expressed as saying that air flow in the Northern hemisphere will move from a high to a low pressure zone with a bend to the right (Crawford, 1978). Show that ***Coriolis*** forces arise because of the spin of the earth and can be expressed as a non-diagonal metric term $g_{t\phi} = \gamma_{tt} A_\phi$, (Ryder, 2009). The vector potential A_ϕ gives a ***characteristic payoff*** $\omega_z = F_{\theta\phi}$ and a velocity ($\mathbf{v}$) force term $\boldsymbol{\omega} \times \mathbf{v}$. Assume points on a sphere are measured in latitude θ and longitude ϕ. Show that this follows from the conservation law Eq. (1.76) with

time as the inactive coordinate. If there is azimuthal symmetry, show that this symmetry produces a ***centripetal acceleration*** $\partial_r \gamma_{\phi\phi}$.

13. Show that the earth's rotations produces forces that represent three dimensional cyclonic flows that rise (increase the distance r from the center of the earth), and move for example from the equator towards the pole, fall and then travel back to the equator. Such flows divide the earth into atmospheric cells that play a critical role in global weather formations (Crawford, 1978). They generate *frame orientation effects* that would not be unexplained if we were unaware of the rotation.