2 Inertial behaviors

We argue that the dynamics of systems is governed by forces, which provide a foundation for our approach based on a consideration of the *active* and *inactive strategic* choices available to each player in making a decision. A second aspect of our approach is that each decision is an act that occupies a point in strategic space-time. ***Strategic occupation*** has particular attributes that are commonly identified: an occupied space is identifiable and may attract behavior towards it or around it. In this chapter we inquire about such ***inertial*** attributes and incorporate them into out theory.

Looking purely at kinematic or ***non-inertial*** aspects, we provided in the last chapter a motivation for incorporating the payoff field of each player into our *decision process theory*. We borrow the term "***inertial***" from the physical sciences: an ***inertial force*** reflects the resistance of a body at rest to move or a body in uniform motion to change direction. In Newtonian physics it is the product of the mass and the acceleration. By extension, an ***inertial frame*** is one in which these attributes of Newtonian physics hold. By contrast, a ***non-inertial frame*** is one in which these attributes don't hold. Two common examples are bodies in a gravitational field and the ***Coriolis effect*** seen by an observer on a spinning globe. A slightly less obvious example is an observer on the inside of the spinning globe feeling the effect of the ***centripetal acceleration***. We argue that ***vorticity*** or spin plays a role in decision making and so we anticipate the importance of non-inertial frames.

We say gravitation is non-inertial because the acceleration of an object is independent of its mass: it reflects the equality of inertial and gravitational mass. In this language, the effects of the payoff field are *non-inertial* and are due to looking at the problem in a *non-inertial frame*: these are ***non-inertial forces***. In *decision process theory*, a system subject only to such *non-inertial forces* follows a geodesic, Eqs. (1.75) and (1.76). It is the generalization of Lagrange's Eq. (1.9) with only the kinetic term $L = T$. In this chapter, we expand *decision process theory* to include ***inertial forces***.

Our reason for expanding the theory is the belief that in its current form, the theory though already an extension of *game theory*, does not yet describe important effects that are clearly part of realistic decision processes. In the business world we are aware of ***inertial forces*** that reflect how hard it is to move some organizations to become more productive, less wasteful, or towards other desirable goals. We are also aware of ***applied forces*** on organizations that reflect quick learners and adopters of best practices because they observe competitors attempting to do the same or similar jobs. Any study of dynamics of necessity deals with two such contradictory mechanisms: one mechanism *staying the course* and the other *making radical changes*. We have borrowed the terms characterizing these behaviors from the study of physical systems and say that there are ***inertial forces*** that provide the resistance to change and ***applied forces*** that induce systems to change. We refer to both as ***inertial behaviors***.

In this chapter we determine the general form for *inertial behaviors*. We find that for *decision process theory*, the general form can be expressed in terms of a tensor that describes the energy and momentum of the system. To obtain this result we first recall Lagrange's Eq. (1.9) and note that it follows from a ***principle of least action***. For ***inertial media*** (such as fluids) that carry *mass*, this principle will give us the mechanism for formulating a general discussion. To get the general result, we include forces that reflect the resistance to change of any decision process as well as forces that reflect learning or *applied forces*. These effects are not included in game theory or in our treatment from the last chapter for non-inertial effects due to active and inactive strategies. The effects we are looking for are attributes of (massive) continuous media. We determine the energy momentum tensor for fluids, an example of (massive) continuous media and then revisit the electromagnetic fields and show that such fields are (massless) continuous media. We then provide a general characterization of *inertial behaviors* in terms of the energy and momentum fields and show the *principle of least action* for such behaviors. We provide a working model for the energy momentum tensor that we use throughout the remainder of this book. We conclude the chapter with advanced topics on some potential limits of our theory based on the limits of the *principle of least action* in physical theories, as well as elaborate on our notion of uncertainty versus frequency.

2.1 *Inertial media*

The characteristics of decision processes that we see missing are the *resistance to change* and the *influence to change* by neighboring processes. This ***inertial cohesion*** is distinct from those influences to change that depend on payoffs and exist without neighboring processes. We call such coupled neighboring processes an ***inertial media***. Financial markets provide an example of such mutual interactions. In this section we investigate the form of such interactions. We borrow from the physical sciences the analysis of continuous *inertial media*. As in the physical sciences, forces generate acceleration. We find a form of the mutual interactions that can be expressed using the Lagrangian formulation Eq. (1.9), which has a geometric interpretation using the ***principle of least action***.

The geometric interpretation arises as follows. Along any path in the conceptual space, the action is defined as the path dependent integration of the Lagrangian times the incremental interval of time:

$$S = \int_{path} L dt \qquad (2.1)$$

There is one path that is special, the one in which the action is an extremum. To find this path imposes a condition on the Lagrangian, which leads identically to Eq. (1.9) and so we are justified in stating that all mechanical systems obey the following ***principle of least action***:

$$\delta S = \delta \int_{path} L dt = 0 \qquad (2.2)$$

The geodesic paths described in the previous chapter are examples of such an extremum. For each of these examples, there is a well-defined action and the minimization of the action is the same as finding the shortest path (geodesic). The *principle of least action* provides an insightful way to view complex systems.

To understand the dynamic behavior of *inertial media*, we note its similarity to ***matter***. *Matter* comes in a variety of forms from gases and fluids to solids such as glasses, metals and crystals. These are continuous systems whose parts mutually interact in a way that is similar to how we see *inertial media* interacting. We approach the study of *matter* by partitioning it conceptually into small *cells*, an approach ascribed (Tolman, 1987) to Laue. We compute the action of each cell and add these together:

$$S = \sum_{cells} \int L_{cell} dt \qquad (2.3)$$

The smaller the cells, the better will be the approximation. The Lagrangian L_{cell} for each cell is determined by the kinetic energy T_{cell} minus the potential energy V_{cell}. It is convenient though not mandatory to consider the cells to be cubes and use continuous variables to describe their essential behaviors.

We do this, starting with the kinetic energy, which is determined by the product of the mass density, the square of the velocity of the cell and the volume of the cell:

$$T_{cell} = \tfrac{1}{2} \rho(x, y, z, t) \dot{\mathbf{r}}_{cell} \bullet \dot{\mathbf{r}}_{cell} dx dy dz \qquad (2.4)$$

We introduce spatial coordinates that represent the center of the cell, a displacement vector $\mathbf{r}(x, y, z, t)$ that represents the relative motion of the fluid (and hence is the dynamic variable for the Lagrangian) at a moment in time t and the average density of the cell at that point and time as $\rho(x, y, z, t)$. If the cell is sufficiently small, the average density is effectively constant over the whole cell at the given moment. Similarly the velocity of the relative displacement at that point is a vector function of the coordinates, $\dot{\mathbf{r}}_{cell} = \mathbf{V}(x, y, z, t)$.

The cell is being compressed and stretched by the surrounding forces as well as turned and rotated. For the *inertial media* of decision processes, the compressions and rotations in strategy space reflect the possible influences on one system due to all the other neighbors. Because the cell is a cube[7], one can obtain the equations of motion from a consideration of the forces on each of the opposing faces. For *inertial media*, this is generalized to a cube in more dimensions.

In three dimensions, there are three forces on the face orthogonal to the $-x$ direction. One of the components is along this direction and the other two forces are transverse along the other coordinate axes. The changes in the forces from one face to its opposite are small and proportional to the area, so we use the characteristic of ***continuous matter*** that there is a ***relative stress*** defined as the force per unit area along that face. For our first case, we name the components of stress $\{-t_{xx} \quad -t_{yx} \quad -t_{zx}\}$ and look to obtain equations that relate the stresses to the rate of change of the momentum.

Along the $+x$ direction, given the distance between faces is dx, the components of stress along the x direction $-t_{xx} - dx\partial_x t_{xx}$ depends on the gradient of the stress; the net stress is therefore (minus) the gradient $-dx\partial_x t_{xx}$. In a similar fashion the other two stress components on this face are $-dx\partial_x t_{yx}$ and $-dx\partial_x t_{zx}$. To get the total force along the x axis we add the component from this face, $-dx\partial_x t_{xx}$, with the face orthogonal to y resulting in $-dy\partial_y t_{xy}$ and the face orthogonal to z resulting in $-dz\partial_z t_{xz}$:

$$f_x dxdydz = -dydz\left(dx\partial_x t_{xx}\right) - dxdz\left(dy\partial_x t_{xy}\right) - dxdy\left(dz\partial_x t_{xz}\right) \tag{2.5}$$

This gives the total force along the x axis due to the forces being exerted from all six faces.

There are similar expressions for the other two components of force, all of which can be compactly summarized as follows, where there is a sum over repeated indices that represent the three coordinate axes:

$$f_j = -\partial_k t_{jk} \tag{2.6}$$

In Newtonian mechanics, the ***inertial force*** is the product of the mass and acceleration and is therefore the time rate of change of the momentum of the cell, which can be given in terms of the momentum density g_j as $g_j dxdydz = \rho V_j dxdydz$. The inertial force is balanced by the net force caused by the stresses $f_j dxdydz$:

$$\frac{d}{dt}\left(g_j dxdydz\right) = f_j dxdydz = -\partial_k t_{jk} dxdydz \tag{2.7}$$

The resultant expression requires that we include the rate of change of volume and then simplify it to provide a general description of the behavior of matter in terms of the momentum density g_j:

$$\frac{\partial g_j}{\partial t} + \partial_k\left(V_k g_j\right) = -\partial_k t_{jk} \tag{2.8}$$

For a continuous media, these are Euler's equations which represent the conservation of momentum: the rate of change of the flow times the matter density is the transverse gradient of the stress.

Note that the generalization to *inertial media* is accomplished by extending the sum so that it goes over the total number of spatial strategic directions: the form of the equations remains the same. We now have the first result in terms of the ***inertial cohesion*** represented by the stresses t_{jk}: changes in the stresses generate changes in the momentum of the system. Since the momentum of the system depends on its inertia, we are justified in our assertion that the applied effects are inertial in character.

The second attribute of fluids is the *conservation of mass*, based on the observed fact that the net matter flowing into a cell causes the amount of matter inside that cell to increase by the same amount:

[7] Space can always be partitioned into cubes. What is important is the partitioning, not the exact shape. If other shapes are considered, it may be necessary to consider more than one such shape to "fill" space.

$$\frac{d\rho}{dt} = -\rho\partial_j V_j \Leftrightarrow \frac{\partial\rho}{\partial t} + V_j \partial_j \rho + \rho\partial_j V_j = 0 \tag{2.9}$$

This result reflects one attribute of the resistance to change of inertial systems. It generalizes to *inertial media* by extending the summation to all the spatial directions.

The momentum density and matter density are thus determined by two coupled equations:

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \partial_j g_j = 0 \\ &\frac{\partial g_j}{\partial t} + \partial_k \left(V_k g_j \right) = -\partial_k t_{jk} \end{aligned} \tag{2.10}$$

These two equations generalized to *inertial media* represent the major result for this section, though not yet written in the most illuminating form. We will write these results using the *principle of least action*. To incorporate these results into *decision process theory*, we will further rewrite them in a fashion consistent with the terminology from the last chapter.

To achieve these goals, we put the equations into a form that combines both the space and time aspects together in a covariant form. We introduce the ***absolute stress***:

$$p_{jk} = t_{jk} + g_j V_k = t_{jk} + \rho V_j V_k \tag{2.11}$$

For a frame in which the media is moving, the stress tensor t_{jk} will in general not be symmetric; however the absolute stress will be symmetric (Tolman, 1987, p. 65ff).

We next rewrite Eq. (2.10) in terms of these absolute stress components:

$$\begin{aligned} &\frac{\partial g_j}{\partial t} + \partial_k p_{jk} = 0 \\ &\frac{\partial\rho}{\partial t} + \partial_j g_j = 0 \end{aligned} \tag{2.12}$$

The second equation shows that the increase in matter is due to the amount of momentum that flows into the cell; the first equation shows that the increase in momentum is due to the amount of stress that flows into the cell.

As an aside, a good idealization for many common materials is that of a perfect fluid, one in which the absolute stress is determined by a single quantity typically called the pressure: $p_{jk} = p\delta_{jk}$, where the "delta" notation indicates that the value is unity whenever the indices are the same and zero otherwise. The above equations then become:

$$\begin{aligned} &\rho\frac{\partial V_j}{\partial t} = \partial_k \left(\rho V_j V_k - p\delta_{jk} \right) \\ &\frac{d\rho}{dt} + \rho\partial_j V_j = 0 \end{aligned} \tag{2.13}$$

At low velocities, the first equation is Euler's equation that the rate of change of momentum is minus the gradient of the pressure. When there are no applied forces, a system in constant motion remains in motion; a system at rest remains at rest. These are the expected effects for inertial forces.

For *decision process theory* if we adopt the concept that inertial effects are characterized by a density ρ that is a continuous function of position and time, we expect to get equations of this type as well. The conservation law (second equation) in Eq. (2.13) expresses the conservation law for inertia, namely the net flow of inertia into a cell determines the rate at which the amount of inertia inside that cell increases.

Returning to the general case, we represent these results with a matrix that adds to space a new dimension proportional to time: $x_0 = ct$. The proportionality constant has units of a velocity so that the new dimension has units of distance. We define the components of a tensor, a field with two indices, T_{ab} as follows:

$$
\begin{aligned}
T_{jk} &= p_{jk} \\
T_{0j} &= -cg_j \\
T_{00} &= \rho c^2 \\
T_{ab} &= T_{ba}
\end{aligned}
\tag{2.14}
$$

The absolute stress, the momentum density and matter density combine to form a symmetric ***energy-momentum stress tensor***. The conservation laws of momentum and matter in this notation are:

$$
\begin{aligned}
\partial_0 T_{j0} - \partial_k T_{jk} &= 0 \\
\partial_0 T_{00} - \partial_j T_{0j} &= 0
\end{aligned}
\tag{2.15}
$$

The component T_{00} has the units of energy per unit volume, an energy density, since the proportionality constant is a velocity and mass time velocity squared has units of energy. Similarly the component T_{0j} has units of energy density since it is a momentum density times a velocity. We now have a form for the *inertial media* expressed in the notation from the previous chapter:

$$
g^{bc}\partial_b T_{ac} = 0 \tag{2.16}
$$

For the general case we therefore characterize ***inertial cohesion*** using a conserved energy momentum tensor. Many systems including non-inertial systems have such conserved tensors. Before examining the important example of payoff fields that also have this form, we reflect on our result.

Newtonian physics starts with the notion that the inertial effects result because bodies have mass. In Newtonian physics, bodies with mass are idealized as ***rigid bodies***. More significantly, Newtonian physics requires *rigid bodies* in order to define the measurement process: lengths are measured using rigid rulers. Nevertheless, within that Newtonian framework it is possible to describe continuous bodies. They are the opposite of rigid bodies, showing that as one part moves, the effect is communicated to neighboring parts through the local continuity of momentum and matter. It is clear that such effects do not move instantaneously, but at a speed characteristic of the medium. By contrast, a *rigid body* communicates its motion instantaneously to all other parts. Our best observations to date indicate that such instantaneous communication does not happen. Thus, matter as we know it, reflects the properties of a continuous medium or field. For this reason we consider inertial effects as being associated with continuous media rather than rigid bodies.

Einstein noted that general relativity is the physical theory that properly describes continuous media, though for consistency it must depart from Newtonian mechanics. In the theory of relativity, all phenomena are described by an energy momentum tensor, reflecting continuous media. In the next section, we give an exemplar phenomena of continuous media—electromagnetic phenomena—that is not associated with mass or inertia. In *decision process theory*, payoff fields are a generalization of such phenomena.

2.2 *Unifying inertial media with payoff fields*

We will write a consistent *principle of least action* for **inertial media** (the generalization of *matter*) and for *payoff fields* (the generalization of *electromagnetic fields*). We saw in the last section that *inertial media* communicate effects with a finite velocity, not instantaneously. This is also a striking aspect of Maxwell's equations and their payoff field generalizations. Information is transmitted from one part of space to another via the electric and magnetic fields at the speed of light as indicated by the wave equations Eq. (1.16). We expect that this will be a general consequence of the *principle of least action* for any type of continuous media.

The action for the electromagnetic field and its interaction with matter is:

$$
S = \int \left(\tfrac{1}{4} F_{ab} F^{ab} + \rho V^a A_a \right) dxdydzdt \tag{2.17}
$$

The extremum for this action yields Maxwell's Eq. (1.24) that includes the interaction of the matter with the electromagnetic field. The major difference between matter and radiation is that the electromagnetic radiation is massless: there is no frame in which light is at rest. Signals move more slowly through matter than they do through radiation. The electromagnetic field behaves like a continuous medium, with its own energy momentum tensor that follows from the above action.

The energy momentum field for the electromagnetic field that results is:

$$T_{ab} = F_{ac}F^{c}{}_{b} - \tfrac{1}{4} g_{ab}F_{cd}F^{cd} \tag{2.18}$$

In the absence of matter, the equations of motion Eq. (1.24) are summarized succinctly by the requirement that the energy-momentum tensor is conserved, an equation identical to Eq. (2.16):

$$g^{bc}\partial_b T_{ac} = 0 \tag{2.19}$$

This tells us that in the absence of matter, the electromagnetic phenomena move in a way to conserve momentum and energy[8]. In this case, the energy is that of the field and has no connection to any rest mass. The energy component T_{00} is $\tfrac{1}{2}\mathbf{E}\cdot\mathbf{E} + \tfrac{1}{2}\mathbf{B}\cdot\mathbf{B}$, the sum of the electric field squared and the magnetic field squared[9]. It is positive and independent of matter. The continuity equation is clearly one of the continuity of flow of energy. Similarly one can compute the momentum for an electromagnetic wave, which is formed from the product of the electric and magnetic fields. The momentum points in a direction orthogonal to both. Thus momentum travels as a wave orthogonal to the electric and magnetic field vibrations and does so as if the vibrations were in a continuous medium.

Since light is a particular type of electromagnetic wave, the study of optics provides a good example of the character of the waves under specialized circumstances in which the wave lengths are small compared to the size of the “lens”. In this case the wave phenomena appear to be more particle-like, with light traveling along paths of least time (which is a special case of least action). If light goes through water it travels slower than going through air and this leads to the well-known bending of light at the boundary. The often quoted example is that of a person on the beach wishing to save someone; since they can run faster than they can swim (normally); it is faster to run towards a point that has less swimming distance, than directly at the person drowning. This optimizes their time to the drowning person. Of course it does raise the question of why light travels at a different speed in matter than in a vacuum. The deeper answer would again require us to have the action for light and matter, where we would then see that light interacts with charged particles (the electrons) and gets absorbed by atoms and re-emitted. The effective interaction is characterized by saying the light moves slower, though the deeper explanation would allow one to compute the amount by which light slows down depending on the properties of matter. The slowing of light is described empirically by assigning to matter an index of refraction that specifies the velocity of light in that medium.

We are now in a position to write a consistent *principle of least action* for ***inertial media*** (the generalization of *matter*) and for *payoff fields* (the generalization of *electromagnetic fields*). The simplest unification is a single conserved energy momentum tensor that contains both fields and is derivable from the *principle of least action*. We give below a more sophisticated unification in terms of the action, Eq. (2.24) and the conservation law Eq. (2.26). The more sophisticated unification requires a change in perspective from the way we are used to thinking about measurements (geometry) and continuity (topology). We assert that this thinking is relevant and necessary for *decision process theory*. To construct a dynamic theory of decision processes, these are precisely the concepts we must articulate clearly. We outline the change of perspective that was needed to describe the physical world and indicate the generalizations needed for our theory.

We recall that the Newtonian view is based on our *common sense* that certain forces, such as gravity and charge are instantaneous. We showed that this view is not consistent however with Maxwell's equations Eq. (1.20) and Eq. (1.24), which show that the Coulomb force Eq. (1.17) propagates with the

[8] The energy-momentum tensor is also conserved in the presence of matter. We would write the total energy-momentum tensor as the sum of the electromagnetic field and the matter field. Each have an energy-momentum that is conserved separately.

[9] An electrical engineer understands that energy is really stored in capacitors as the electrical field and in inductors as the magnetic field. These fields are observed to be real and often quite powerful.

velocity of light. The Newtonian view also maintains the *common sense* view of the existence of *rigid bodies*, which have the same defect of propagating effects instantly from one part of the body to another. However, we have suggested that real bodies are continuous and communicate from one part of the body to another at a finite speed. We were thus led to describe inertial effects by means of ***inertial media***. Einstein unified these viewpoints into a theory of relativity in which no instantaneous transmission of forces exists and in which there are no *rigid bodies*. We use his new *common sense* view as basis for our general *decision process theory*

To appreciate the philosophical change in perspective that Einstein requires of us, we discuss geometry and the measurement of distance. Measurements are made using some type of ruler. They presuppose an understanding of the space in which the measurements take place. This space already has some basic topology that distinguishes near from far. At this level there is no distinction between a coffee cup and a donut: both are spaces with a hole in them. We distinguish such objects when we provide a measure of distance between neighboring points. Euclidean geometry in three dimensions is based on the ***distance*** between two points being given by

$$\Delta s = \sqrt{\left(x_A - x_B\right)^2 + \left(y_A - y_B\right)^2 + \left(z_A - z_B\right)^2} \tag{2.20}$$

The coordinates reflect the three dimensional nature of space—the x, y and z axes of space which are characterized as being orthogonal. Euclidean geometry further characterizes space as flat, so that the above formula holds whatever the distance between the two points. The ruler is assumed to be a rigid body whose existence is not part of the dynamics: it measures the same at every point in space. Thus it is enough to know the coordinates of the two points in order to determine their distance.

In Newtonian physics, if the two points represent successive positions of a body that is moving, then the distance depends on the time difference of the two points. If the two points are extremely close together in time then Newtonian physics observes that the distance is proportional to time; the proportionality constant is the ***speed*** and the ***velocity*** is a vector in the direction of motion of the body at that point in time, whose magnitude is the speed. This is a non-trivial statement about how measurements occur in time. Many first year students in physics and engineering in their physics' lab course take a series of photographs of rapidly moving objects as a way to "see" the speed. They observe that the speed as defined as the ratio of distance interval over time interval is a number independent of the time interval, once the time interval is made small enough. The ideas of distance, velocity and the rate of change of velocity (called ***acceleration***) are sufficient distinctions with which to describe the physical laws in Newtonian physics.

We suggest that for dynamics of any system, there are in fact two attributes: the topology and the metric. We don't give up the notion of measuring, only the notion, following Einstein, that the measuring sticks are rigid and outside of the dynamic equations. Rigidity is a good approximation for points that are neighbors, but not for points that are well separated. It is plausible therefore that we should return briefly to the notion of the underlying topology and inquire into the nature of what we mean by local.

An example may be helpful. We are well aware that the surface of the earth is a (oblate) sphere. From a topological point of view, there are two dimensions to the surface of the earth, whatever its shape: this is a topological property. Moreover, the sphere has no "holes", another topological property. The sphere has very interesting local properties however. At any point, there is a continuous, differentiable and one-to-one mapping of the sphere onto a plane. In normal parlance we call such a mapping a chart and use such charts for navigation purposes. We navigate with rigid rulers on the charts, despite the fact that we are plotting points on a sphere. The differentiable map, or ***diffeomorphism,*** is an exact statement only at the point and a small region around it. We know that the charts lose accuracy away from that point, so when we travel we need a number of charts in order to navigate around the earth. The charts have to match on their common areas. It is possible to describe the sphere in this fashion as a series of two-dimensional charts. The collection of all the charts will then be our atlas. The structure defined in this way is termed a ***differential geometry*** by mathematicians and used by physicists such as (Hawking & Ellis, 1973). It is also studied in a branch of mathematics called fiber bundles (Steenrod, 1951). It provides a precise language with which to describe all the attributes of such structures.

The differential geometry makes only the assumptions at the point that reflect our direct experience. At the given point we know that any rotation of the two dimensions yields an equivalent set of charts. In order to navigate between points we need to know the symmetry structure at each point and we need to know how to glue the charts together, which is tantamount to our knowing how a path appears on each chart as it moves over the surface of the earth so that the charts provide a faithful representation of the physical geometry. To apply this to any system, we have to identify the dimensions in which we live, our topological space, identify the local symmetries that are present for all dynamics and identify the charts that are needed to describe motion. For *decision process theory*, we generalize Einstein's approach to identify the topology as consisting of strategic space and time.

Einstein takes the implied form of the metric from Maxwell's equations. Consider an electromagnetic wave that satisfies the wave equation Eq. (1.16). The waves are real numbers, but are usefully thought of as a superposition of complex numbers:

$$\begin{aligned} E_a &= c_{a\mathbf{k}} e^{i\omega t + i\mathbf{k}\cdot\mathbf{r}} \\ B_a &= d_{a\mathbf{k}} e^{i\omega t + i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{2.21}$$

Electrical Engineers use these ***phasors*** to simplify calculation and take real parts at the end. The electric and magnetic fields for radiation are computed as the real or imaginary parts of this superposition. Based on the wave equation, the relationship between the frequency ω and the wave number vector $\mathbf{k}$ is:

$$\omega^2 - c^2\mathbf{k}\cdot\mathbf{k} = 0 \tag{2.22}$$

This expresses the fact that the wave travels with a constant speed c called the speed of light. Einstein identifies a 4-vector as one in which the time component is the frequency ω and the space components are the wave numbers $c\mathbf{k}$. With this identification the length of the 4-vector is determined by the metric we suggested earlier, Eq. (1.23).

Though the Newtonian view and Maxwellian views are different, it is possible that both are correct in their domain of application. However there are phenomena that require both theories as part of the explanation. The speed of light from a moving object should increase or decrease its speed according to the Newtonian view; the energy (frequency) of the light should increase, but the speed should stay the same according to the Maxwellian view. Michelson and Morley demonstrated that the speed in fact stays the same, showing that the Maxwellian view is more accurate. The effect they observed was very small, so that no known deviations to Newtonian mechanics would have been observed. Einstein's method was to provide a theory of mechanics consistent with the Maxwellian view. He took as the observational fact the form of the metric as used in Maxwell's equation. Based on what was understood about electromagnetic phenomena, he postulated that the descriptions of all physical phenomena would have the same form in any frame of reference as long as one understands that measurements are done using this metric, called a ***Lorentz metric*** in physics after the physicist who first articulated its importance:

$$\Delta s = \sqrt{c^2\left(t_b - t_a\right)^2 - \left(x_b - x_a\right)^2 - \left(y_b - y_a\right)^2 - \left(z_b - z_a\right)^2} \tag{2.23}$$

This is the answer to the question: what is the topological space? It is a 4-dimensional manifold, diffeomorphic at each point to a pseudo-Euclidean (Minkowski) space with the metric Eq. (2.23). Transformations that leave this form invariant also leave the form of the electromagnetic action Eq. (2.17) invariant when expressed in terms of the metric. Einstein conjectured that in general, at each point the action would be invariant under these transformations, thus supporting the idea that space is a 4-dimensional continuum of values. This conjecture requires changes to the Newtonian laws and in fact to the form of Maxwell's laws in areas where the space is not flat.

The use of a Lorentz metric to measure distance changes subtly, though profoundly, the concept of time, which is related to the fact that energy and momentum densities move with a finite velocity. Newton saw time as a parameter that had the same value at all parts of the universe. Einstein said there was no experimental justification for that assumption; when we use time we always in fact observe only the local time. If we gave someone a watch and told them to tell us what time it was at some distant location, they would have to send us a signal (telephone, radio wave, etc.) that would take a finite amount of time to reach us and we would know at our local time what they believe is their local time. We still would not know their time simultaneously with a measurement of our local time. In other words, we only know the time relative to our chart; a distant person will use a different chart and we will have to learn how to match up the charts.

Second, Einstein asserted that time is a quality just like space and the two form the conceptual space-time for physical events. Space and time are different from the topological positions; they have a physical reality. In particular, measurement of distances between two events is now a quadratic form that involves time. It creates a space that is non-Euclidean because the measurement of neighboring points is not the sum of squares of the distances.

The third change of Einstein was to formulate the laws of motion in a way consistent with this new way to measure distance and consistent with a new set of mechanical laws based on the same principle of form that Maxwell's equations satisfied. Maxwell's equations Eq. (1.24) have the property that the form of the equations is the same (***covariant***) in every ***frame of reference***, irrespective of the relative motion of the two frames. A special case of this is that the speed of light is always a constant in every frame.

Einstein's conceptual space provides an ordering and a topology (concept of continuity) as well as fields that describe matter, radiation (such as the electromagnetic radiation) and what is new, a physical field for measuring distances (at least in a holonomic coordinate basis) called the ***metric*** g_{ab} that carries energy and momentum. It determines the distance between points $g_{ab}dx^a dx^b$ as the product of the metric and the product of two coordinate distances, summed over all such product possibilities. The continuous metric (media) replaces the rigid measuring rod. Two observers moving relative to each other will use different metrics to describe the same physical process. At any point in space and any instant of time, it is always possible however to find a frame of reference in which the metric is flat, Eq. (2.23); however at some other point and some other point in time, this observer will in general not see the same flat metric.

2.3 *Foundational results*

We conclude from the previous section the foundational results that unify inertial fields with the payoff processes of game theory. Since *rigid bodies* do not exist in the Einstein view, we say that matter (inertia) is described in terms of fields. Such inertial fields are determined by the Lagrange *principle of least action* in terms of the Lagrangian density, where we integrate over space and time, analogous to the examples we have given earlier for the Maxwell fields:

$$S = \int \mathcal{L}\sqrt{g}\,dxdydzdt \qquad (2.24)$$

This form for the Lagrangian can be used to specify a form for matter that is continuous and consistent with Maxwell's equations and our new *common sense*. For *decision process theory*, this suggests that we take seriously not only the principle that flow is influenced by the payoff fields Eq. (1.75) but that we apply the generalized version of Maxwell's equations Eq. (1.24) to decision processes: decision flows are the sources for the payoff fields. We say more about the exact form of the generalization in the next section. Here we clarify the relationship between the inertial fields and the action.

We require the form for the Lagrangian in Eq. (2.24) to be covariant in terms of the metric. First we assume that the Lagrangian density $\mathcal{L}$ is a scalar, *i.e.* it has the same value in every frame of reference. The action is a scalar, so we require the unit of volume to be a scalar. So for example if each coordinate were replaced by itself multiplied by the constant λ, the volume $dxdydzdt$ would be increased by λ^4. Thus the volume is not a scalar quantity. An invariant unit of volume is obtained using the fact that the new metric, will have elements that are λ^{-2} times the old elements; in the special case that the metric is diagonal, the square root of the product of the diagonal elements, which we call $\sqrt{g}$, is λ^{-4}. Thus the product $\sqrt{g}\,dxdydzdt$ is a scalar function and has the same form in every frame. We call g the determinant of the metric viewed as a matrix and show that the above argument holds for all transformations and all possible forms for the metric.

The action for the electromagnetic fields Eq. (2.17) also can be put in a form valid in any frame by correcting the volume element. The Lagrangian density is already invariant. Moreover, for *inertia media*, whose equations of motion follow from the *principle of least action* using the Lagrange form above, we can uniquely define the energy momentum stress tensor by considering the variation of the action δS with respect to changes in the metric δg_{ab}:

$$\delta S = \int T^{ab} \delta g_{ab} \tag{2.25}$$

The energy momentum tensor so defined will be a symmetric field, expressible in terms of the *inertia media* fields and will satisfy the conservation/continuity Eq. (2.15) that is now written covariantly, Eq. (1.52), in terms of the metric:

$$g^{bc} T_{ab;c} = 0 \tag{2.26}$$

Thus we recover and generalize the description given earlier for the fluid example of continuous matter, we obtain a description that is consistent with the observations of Maxwell for charged continuous matter and we provide a covariant description whose equations are generated by the *principle of least action*. We thus accomplish our goal of incorporating *inertial resistance* and *inertial cohesive* forces into our *decision process theory* through the least action principle Eq. (2.25). We specify possible models by specifying possible *inertial media* contributions to the energy momentum tensors T_{ab}.

Following Einstein then we take the general *decision process theory* from the last chapter and add to it the possibility that there are *inertial media* (matter) fields that provide ***inertial forces*** that can be specified by an energy momentum tensor T_{ab}. However, the measurement part due to the metric is new. We have added non-inertial orientation potential fields that are closely related to the metric, Eq. (1.58), which provide for measurements. With all of these changes, our *decision process theory* will be internally consistent. We need to look more closely at how the orientation potentials generate fields Eq. (1.67), the ***orientation flux field*** and how they carry energy and momentum. We have already seen one aspect of that by demonstrating that the payoff fields are specific components of these orientation potentials, Eq. (1.74). Einstein proposed and we adopt the same general rule that we must have dynamic equations that unify not only the *inertial media* and *payoff fields*, but also the orientation flux fields. We do that in the next section.

2.4 *Including orientation flux fields*

We have proposed a foundation for decision process theory in the last section, with an implied set of equations that follow from the *principle of least action*. In this general sense we follow a strategy consistent with that followed in the physical sciences. Two new additions to game theory are: the idea that the payoff fields are generated by the decision flow; and the existence of inertial fields that influence the dynamics. The first addition implies that there is a relationship between some components of the metric fields as determined by the payoff potentials A_a and the sources. We find in this section that this is a consequence of a more general relationship proposed by Einstein for physical properties: the metric is determined by the inertial forces through the *principle of least action* applied to the metric fields. We make that argument below with the generalization to *decision process theory* in mind.

As part of ***decision process theory***, we have identified a metric that provides a measure for decision events. Such a measure however is not outside of the theory but an integral part of the theory. Einstein observed that measurements are physical processes and we make the assumption that for decision processes, measurements are a type of decision process. Since measurements are constrained by the orientation potentials that we described in section 1.9, these quantities must be determined also by a *principle of least action*. These considerations lead us to the equations for *decision process theory*, Eq. (2.31), which follows from the *principle of least action*.

The potentials for electromagnetism carry energy and momentum, implying no instantaneous action at a distance. The requirement for the orientation potentials must be the same. The analogous field Eq. (1.67) is the generalization of Stokes' theorem for a non-commuting gauge group:

$$\Re = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} \tag{2.27}$$

Stokes' theorem relates the total field through a bounded area to the sum of differential quantities around the boundary. For a general group, in which some elements don't commute, the total field through an area bounded by a closed loop equals the covariant differential of the potential integrated around the loop. At a deep mathematical level, the orientation flux field above is this covariant differential.

For the electromagnetic field the second term of Eq. (2.27) is zero. This is not the case for the group of frames, the symmetry group associated with the orientation potentials. Ordinarily, the ***orientation flux field*** is called the curvature tensor of the space and is written as $\mathbf{R}$. The notation Eq. (2.27) is compact: the curvature is a matrix of 2-forms that measures the flow or change of the orientation of a frame through a 2-surface into the dual of a second 2-surface:

$$\mathfrak{R}^a{}_b = R^a{}_{bcd}\mathbf{E}^c \wedge \mathbf{E}^d \tag{2.28}$$

The components, the tensor $R^a{}_{bcd}$ are antisymmetric in the last two indices. The tensor is also antisymmetric in the first two indices. By Eq. (2.27), the curvature tensor components are determined by the *orientation potentials*.

Since we have an equation of action for the electromagnetic field in terms of the field strength $\mathbf{F}$, we expect a similar principle to hold for the curvature. Of the many forms that have been found and are equivalent, the following is one that depends directly on the curvature 2-forms, and physically represents the flow of the *orientation flux field*:

$$S = \int \mathfrak{R}^a{}_b \wedge *\left(\mathbf{E}_a \wedge \mathbf{E}^b\right) \tag{2.29}$$

The curvature 2-form is multiplied by a 2-form constructed from the "dual" or "$*$" operator of $\mathbf{E}_a \wedge \mathbf{E}^b$, defined for a general n-dimensional space as the $n-2$-form:

$$*\left(\mathbf{E}_a \wedge \mathbf{E}_b\right) = \varepsilon_{abc\cdots d}\mathbf{E}^c \wedge \cdots \wedge \mathbf{E}^d \tag{2.30}$$

In four dimensions this is constructed from the tensor ε_{abcd} that is antisymmetric in every pair of indices and has the value +1 for ε_{0xyz}. It can be used to evaluate a determinant and so in fact is used to define the volume element that is written above as the square root of the determinant of the metric $\sqrt{g}$ times a 4-form. The Lagrangian density is thus the wedge product of two 2-forms and will be an invariant under all frame transformations. It is also a scalar under gauge transformations, so the resultant theory is a gauge invariant theory. We require the generalization of this to any number of dimensions.

The evidence for the *principle of least action* using Eq. (2.29) is the Newtonian law of gravity. In Einstein's view, gravity is not an instantaneous force, but one that propagates with the speed of light from one point of space to another, preserving continuity of the energy and momentum stress densities. He showed that the various attributes of Newtonian theory were consequences of this theory, as well as small but observable changes to the Newtonian view. One consequence is that gravity is not a static scalar field as it is described by the orientation potentials whose components change dynamically with time.

The result of the *principle of least action* with Eq. (2.29) extended to include *inertial media* fields is shown (Gockeler & Schucker, 1987) to result in the following expression:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = -\kappa T_{ab} \tag{2.31}$$

In evaluating the least action, the full curvature tensor does not appear, only two of its contracted forms, $R_{ab} = R_{dabc}g^{cd}$ and $R = g^{ab}R_{ab}$. The right hand side is the product of the energy-momentum tensor Eq. (2.25) of the *inertial media* fields and a universal constant κ. The equations are gauge invariant and frame covariant. A subset of these *decision process theory* equations provides the generalization of Maxwell's equations as shown in Section 3.11, exercise 10.

Energy and momentum are conserved using covariant derivatives for the *inertial media* field and for the orientation flux field separately:

$$g^{bc}T_{ab;c} = 0 \tag{2.32}$$

Compare this with our general expressions for the stress tensor for matter, Eq. (2.19). Here we have the ***covariance principle*** that in describing rates of change, ordinary derivatives are replaced by covariant derivatives Eq. (1.52).

We have the last major piece of our *decision process theory* from the consequence of Eq. (2.31). The unified applications of the *principle of least action* to all of the fields that participate in the decision process bring together the *orientation flux fields* and the *inertial media* fields. The *inertial media* fields reflect both inertial and application forces. All of these fields carry energy and momentum. All of these fields are real. All of these fields obey the conservation laws of energy and momentum Eq. (2.32). Our last result for this chapter will be to investigate useful models for the *inertial media* field, which we do in the next section. First however we summarize a few results for reference later in the book.

We consider some of the consequences of the general form of the Einstein equations Eq. (2.31). In general, the space will not be flat relative to a given set of orientation potentials. Energy and momentum create the source for the orientation potentials, which themselves create a flux that carries energy and momentum. So although we argue that the space is locally flat, the existence of non-trivial fields changes how individual charts are pasted together to describe a more complex global structure.

We have argued that this is no different from looking at the earth in terms of flat charts locally, yet nevertheless the space is curved. The curvature is a consequence of how the charts must be glued together; it is a consequence of the atlas. We measure effects of local curvature when we look at second order rates of change. For example, we envision this process by looking at a small square with sides dx^a and dy^a. We take a vector V^a and displace it first along the direction dx^a and then along dy^a; alternatively we go first along dy^a and then along dx^a. The difference of the two calculations can be evaluated in terms of the covariant derivatives:

$$\left(V^a{}_{;bc} - V^a{}_{;cb}\right) dx^b dy^c = R^a{}_{dcb} V^d dx^b dy^c \tag{2.33}$$

We make the reasonable statement that the difference between the two directions will be a set of numbers $R^a{}_{bcd}$ that provide the proportionality constants multiplying the area of the square and proportional to the size of the vector being translated. This is again a statement of Stokes' theorem and is a restatement of the arguments that led to Eq. (1.67), so that the numbers $R^a{}_{bcd}$ are the components of the orientation flux tensor. The theorem from geometry is that a necessary and sufficient condition for the space to be flat is that the curvature components vanish everywhere.

For detailed calculations, we start with the components of the curvature tensor given in terms of the orientation potentials, Eq. (2.27):

$$R^a{}_{bcd} = \partial_c \omega^a{}_{bd} - \partial_d \omega^a{}_{bc} + \omega^a{}_{ce} \omega^e{}_{bd} - \omega^a{}_{de} \omega^e{}_{bc} \tag{2.34}$$

The reduced tensor is:

$$R_{ab} = g^{cd} R_{cabd} \tag{2.35}$$

It is formed by contracting two of the indices with the metric and so it is by construction a tensor that is second order in the derivatives of the metric. This matrix and the scalar $R = g^{ab} R_{ab}$ formed by further contracting the tensor with the metric are the quantities that determine the equations of motion from the action Eq. (2.29). The curvature tensor R_{ab} can be evaluated in terms of the orientation potentials. For the special case of a holonomic coordinate basis, the orientation potentials are determined by the metric using Eq. (1.59) and the reduced curvature tensor is:

$$R_{ab} = \partial_a \partial_b \ln \sqrt{|g|} - \partial_c \omega^c_{ab} - \omega^c_{ab} \partial_c \ln \sqrt{|g|} + \omega^d_{ac} \omega^c_{bd} \tag{2.36}$$

These components are second order derivatives of the metric and generate a "wave nature" of the solution when sources are absent and are related to the sources that are present by the *principle of least action* Eq. (2.31).

2.5 *Preliminary model for the energy momentum tensor* T_{ab}

The *decision process theory* includes the possibility of *inertial media* fields that reflects all *inertial forces* that are essential to decisions. We also have the *non-inertial* contributions that are determined by the orientation potentials. And the two sets of fields satisfy the *principle of least action* that leads to an inter-relationship of *inertial* and *orientation flux fields* Eq. (2.31). This is a covariant expression, so is the same in every frame. It relies on theorems from differential geometry that the orientation flux tensor (curvature tensor) is in fact a tensor field. The reduced tensors are also tensor fields. Specific models then depend on studying possible forms for the energy momentum tensor T_{ab}. We start with the form we derived for a fluid, Eq. (2.14):

$$T_{ab} = \mu V_a V_b - p_{ab} \tag{2.37}$$

To start we consider this expression in the ***co-moving coordinate basis*** that moves with the fluid. We introduce a vector field V^a to describe the flow of the fluid and assume that it has unit length:

$$g_{ab} V^a V^b = 1 \tag{2.38}$$

In the co-moving basis this determines the time component of the vector field in terms of g_{00}. From our previous work, the stress components p_{00} and p_{0b} are zero. The time component of the energy momentum tensor is the energy density $T_{00} = \mu V_0 V_0$.

If the basis is not only co-moving but Minkowski, the energy density is equal to the time component of the stress tensor. In this same basis, there are no momentum components and the space components of the energy momentum tensor are given by p_{ab}. Therefore we call the latter the ***stress tensor components*** and in a general frame impose the conditions:

$$\begin{aligned} p_{ab} &= p_{ba} \\ p_{ab} V^b &= 0 \end{aligned} \tag{2.39}$$

We have defined a fairly general model. The energy momentum is a symmetric tensor that will in general have real eigenvalues at each point, one of which will have a positive length μ; the remaining will have negative values. The eigenvector corresponding to the positive eigenvalue defines the vector field V^a.

We can then look at the difference $T_{ab} - \mu V_a V_b$ which we define as $-p_{ab}$. It has no positive eigenvalue, but only negative eigenvalues that are projected onto the space orthogonal to the flow by the projection operator:

$$h_{ab} = g_{ab} - V_a V_b \tag{2.40}$$

We recover the general form Eq. (2.37). We thus have a general form we can consider as part of our *decision process theory*. It is useful to define ***pressure*** as the average stress, with n space components:

$$p = \tfrac{1}{n} h^{ab} p_{ab} \tag{2.41}$$

We remain consistent with our unified *principle of least action* Eq. (2.31) as long as our energy momentum tensor components are conserved according to Eq. (2.32). We provide the result of the conservation laws:

$$\begin{aligned} \left(\mu h^{ab} + p^{ab}\right)\frac{DV_b}{\partial \tau} &= h^a{}_c h^b{}_d \, p^{cd}{}_{;b} \\ \frac{d\mu}{d\tau} + \mu V^a{}_{;a} + V_{a;b} p^{ab} &= 0 \end{aligned} \tag{2.42}$$

We have the inertial and application forces expressed in Eq. (2.42): the gradient of the stresses provides a source for the application forces; there is a matter conservation law for the inertial matter density.

For reference a particularly simple model is the ***elastic perfect fluid*** Eq. (2.14) written in covariant form:

$$T_{ab} = \left(\mu + p\right) V_a V_b - p g_{ab} \tag{2.43}$$

This corresponds to the diagonal stress tensor whose elements are equal to the pressure:

$$p_{ab} = ph_{ab} \tag{2.44}$$

The conservation laws are:

$$\begin{aligned}(\mu + p)\frac{DV^a}{\partial\tau} &= h^{ab}p_{;b}\\ \frac{d\mu}{d\tau} + V^a{}_{;a}(\mu + p) &= 0\end{aligned} \tag{2.45}$$

The first equation is Euler's equation and the second is the conservation of *inertial media*, defining the *inertial media* density ρ in terms of the energy density and pressure:

$$\frac{d\rho}{\rho} = \frac{d\mu}{\mu + p} \tag{2.46}$$

The fluid is considered ***elastic*** if the energy density is a function only of the pressure, in which case this equation provides a differential equation for the *inertial media* density in terms of the energy density.

2.6 *Hamiltonian formulation*[10]

Our theoretical foundation relies heavily on the physical *principle of least action*. We need to operate in a domain where that principle makes sense. To explore thus issue, we study the corresponding issue in physics. As part of classical physics (Goldstein, 1959), Hamilton noted in 1834 that surfaces of constant action (Hamilton's principle function) propagate as if they were a wave. Since the Newtonian view articulated in Hamilton's language is the *principle of least action*, a surface of constant action is a mathematical construct. Taking this construct, he argued that it is possible to view classical mechanics as the geometric optics approximation to a wave equation. The equation he obtained is very close to Schrödinger's equation of quantum mechanics. Moreover Schrödinger's equation describes a continuous system; it provides yet another continuous system in which the energy momentum stress tensor is conserved.

We start with some general background and summary of Hamilton's result. Using a slightly different but equivalent variation in which the energy is conserved, the classical motion is determined by Hamilton's *principle of least action* written in terms of the kinetic energy T :

$$\Delta\int T dt = 0 \tag{2.47}$$

The integral provides the historical definition of action[11]. Since the kinetic energy is always a quadratic form in terms of the generalized coordinates, $T = \sum m_{jk}\dot{q}_j\dot{q}_k$, this variation principle can be written in terms of a "path length" $ds^2 = \sum m_{jk} dq_j dq_k$ in the geometry of the active coordinates:

$$\Delta\int \sqrt{2mT} ds = 0 \qquad (2.48).$$

Hamilton's surprising observation is that this form of least action looks like the principle of least time in geometric optics.

Geometric optics is the field of study in which it is assumed that light travels in straight lines like particles whose speed is determined by the index of refraction n:

$$\Delta\int n ds = 0 \tag{2.49}$$

The index of refraction is the inverse of the speed of light through a media, so geometric optics is also formulated as stating light goes from one point to the other in the least time. This simple idea explains why light bends when it enters water, because the index of refraction of water is different from that of air. It provides the theoretical foundations for the construction of optical equipment. Hamilton's *principle of*

[10] This and the next section are advanced. Though their results set the context for our theoretical foundation, they are not needed for engineering calculations of decisions.

[11] Goldstein (Goldstein, 1959) notes that this *principle of least action* is associated with the name of Maupertuis (1747), though the objective formulation of the principle is due to Lagrange and Euler.

least action has the same mathematical form as geometric optics with an index of refraction $\sqrt{2mT}$ determined by the kinetic energy. If there are no external forces (other than possible constraint forces), then Eq. (2.48) shows that the path will be a geodesic (extremum) in the space of generalized coordinates. Equation (2.47) shows that the path will be the one that takes the least time. In order to relate these ideas to action we have to go deeper into the equations of motion.

Both Hamilton and Lagrange hoped to write equations of motion that contained only the essential ingredients required for the solution. The constraints were one thing that could be eliminated. The other thing to identify was the set of variables that were constant in time: these are associated with symmetries of the problem. Even though it might be true that motion along a surface depends on the position, the force might not depend on that position. The earth is a sphere to a high degree of approximation. The gravitational force does not depend on the position on the surface, just on the height. Hamilton's approach to this problem was to consider a function that he could identify with the total energy of the system, and that depends only on the momentum and position. So for example if there were only a single independent coordinate, Hamilton introduced the "Hamiltonian" that is related to the Lagrangian (section 1.3):

$$H\left(p,q,t\right)=p\dot{q}-L\left(q,\dot{q},t\right) \tag{2.50}$$

In conservative systems, the Hamiltonian $H\left(p,q,t\right)$ is conserved and equals the total energy of the system. The first order equations of motion expressed in terms of the Hamiltonian determine the rate of change of the momentum and the position respectively:

$$\dot{p}=-\frac{\partial H}{\partial q},\quad \dot{q}=\frac{\partial H}{\partial p} \tag{2.51}$$

The Hamiltonian formulation makes clear the distinction of ***symmetry*** by noting that if the Hamiltonian is translation invariant along q, then the conjugate momentum p does not change in time. Here by translation invariance we mean that if the value q is changed by a constant amount to $q+a$, then the Hamiltonian is unchanged. Another way to express this is to say that the momentum p is ***conserved***. There is a similar statement in the Lagrange formulation, since in that case the equations of motion are:

$$\dot{p}=\frac{\partial L}{\partial q},\quad p=\frac{\partial L}{\partial \dot{q}} \tag{2.52}$$

If the Hamiltonian is independent of the coordinate, then the Lagrangian is also independent, because of Eq. (2.50).

The Hamiltonian formulation is symmetric with respect to coordinate and momentum since if the Hamiltonian is independent of the momentum, then the rate of change of the coordinate is zero. Momentum and coordinate are truly conjugate variables. We show how this applies to the observation about geometric optics. Hamilton hoped to reduce the solution of every problem to one in which the Hamiltonian was independent of both the momentum and the coordinate. The problem of solving mechanics would then be reduced to transforming variables so that the only unknowns would be the initial conditions. Of course there would be some difficult (partial differential) equations to solve to get to this point. Nevertheless the approach would yield insight into the equations, and from the equations insight into the physical phenomena they describe.

Hamilton defined (contact) transformations that generate new and equivalent Hamiltonian functions that have no coordinate or momentum dependence. He defined a contact transformation as a function $S\left(q,P,t\right)$ that transforms the original coordinate $q=q\left(Q,P,t\right)$ and momentum $p=p\left(Q,P,t\right)$ to a new coordinate Q and momentum P:

$$p=\frac{\partial S}{\partial q},\quad Q=\frac{\partial S}{\partial P},\quad K=H+\frac{\partial S}{\partial t} \tag{2.53}$$

The new Hamiltonian K corresponding to the new coordinate and momentum will satisfy the same type of relationships with the coordinate and momentum as did the old relative to their Hamiltonian:

$$\dot{P} = -\frac{\partial K}{\partial Q}, \quad \dot{Q} = \frac{\partial K}{\partial P} \tag{2.54}$$

Setting the transformed Hamiltonian to zero (depends on no variables) determines an equation for the contact transformation called the Hamilton-Jacobi equation, and is a Hamiltonian in which all the momenta and coordinates are conserved:

$$K(Q,P,t) = H\left(\frac{\partial S}{\partial q}, q, t\right) + \frac{\partial S}{\partial t} = 0 \tag{2.55}$$

This contact transformation is the action S:

$$\frac{dS}{dt} = L \tag{2.56}$$

The equation that determines the transformed Hamiltonian is a partial differential equation for the action.

Ordinarily, we look only at the paths of least action, and not on the value that action takes on other paths. What Hamilton discovered was that if you looked at the action as a function determined from the above partial differential equation, you would see that it describes a (mathematical) wave front determined by surfaces of constant action. To see this, the equation is separated into an energy term and a term defined by Hamilton's "characteristic" function W to determine the wave velocity u of the front:

$$S = W - Et \tag{2.57}$$

In terms of the characteristic function, Eq. (2.55) is:

$$H\left(q_k, \frac{\partial W}{\partial q_k}\right) = E \tag{2.58}$$

A surface of constant action is one in which changes in W in time must be proportional to the energy E, or changes along the surface at the same time must be proportional to the gradient $\left|\partial_k W\right|$ and be zero. If there are changes in both time and distance, the changes in W must be proportional to the path length ds orthogonal to the surface and proportional to the spatial gradient $\left|\partial_k W\right|$. These considerations determine the wave front velocity u:

$$\begin{gathered} dW = Edt = \left|\partial_k W\right| ds \\ u \equiv \frac{ds}{dt} = \frac{E}{\left|\partial_k W\right|} = \frac{E}{\sqrt{2mT}} \end{gathered} \tag{2.59}$$

Hamilton's variation principle Eq. (2.48) can be written in terms of the wave velocity showing explicitly that the path taken is that given by the least time:

$$\Delta \int \frac{ds}{u} = 0 \tag{2.60}$$

The result is general, holding in the presence of external forces.

This deep dive into physics illustrates that the *principle of least action* may be an approximation. The concept of geometric optics, which Hamilton showed describes mechanics, is known to be an approximation. We know that if the wave length of light is large compared to the optical device the wave approach must be taken into account to get an accurate description. Hamilton noted the same thing; if Eq. (2.55) had a small term on the right hand side, then the classical solutions would deviate from Newtonian mechanics, and the paths would no longer be paths of least action in terms of the classical variables. The result is related to our ***decision process theory*** for decisions in that we must also view the *principle of least action* as approximate. We alluded to this problem in section 1.6 in our discussion of units of time. We argued that we must consider times large compared to the characteristics of the process in order to insure that a decision was in fact made.

We suggest that the way to set units of time is to identify a ***quantum of action*** that describes that decision process. Knowing the scale set by the quantum of action would set the units of measuring time and would augment the ***natural units*** defined in section 1.6. In the next section we indicate how quantum effects are introduced in physics, which suggests how one might extend our theoretical foundation in the region in which the *principle of least action* breaks down.

2.7 *Quantum of action*[12]

To understand why least action breaks down, we need to understand where Newtonian physics fails to describe experimental phenomena. We also need to see that the deterministic principle still holds, as well as understand the virtue of frequency versus probability. The first indications of a breakdown were that Newtonian theory deviates from experiment at atomic levels. At atomic levels, a good approximation to observations was provided by Schrödinger, who provided a deterministic continuum model for electrons as fields—a complex number $\psi(x, y, z, t)$ called the wave function describes the electron field:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\partial_k \partial_k \psi + V\psi \tag{2.61}$$

The most important aspect of this equation is Planck's constant, $\hbar$, which has dimensions of action and sets the scale distinguishing classical behavior from quantum behavior. For decision process we suggest introducing a similar constant.

The equation has other attributes worth noting. The potential energy V specifies the forces acting on the electron. Typically it has the same form as the Newtonian potential for a given type of force: Coulomb attraction will be a potential whose strength is inversely proportional to the distance; thus the gradient will give the inverse square law. The electron is described by a continuous field called the wave function that is a complex function of space and time and has a behavior that follows our expectations for a continuous matter distribution. Recall that a complex number $z = a + ib$ is expressed in terms of $i = \sqrt{-1}$, whose square is negative one, or equivalently $z = re^{i\phi}$ in polar coordinates of modulus r and phase ϕ.

The position of the electron at any given time is provided by the absolute value squared of the wave function. In the case described here, the phase of the wave function is not observed. Some people think of this as the probability that the electron will be at a particular position. However, the square of the wave function is not a probability. It would be more accurate to say that it represents the frequency with which you would measure something at that point. The distinction is important since the wave function here obeys a deterministic equation; it is really a wave not a "particle".

The predictions of this theory differ significantly from Newtonian physics only when the action is small compared to Planck's constant. We demonstrate this by writing the complex number explicitly in terms of the phase S and the modulus $\sqrt{\rho}$ that we identify with the matter distribution of the electron:

$$\psi = \rho^{1/2} \exp\left(\frac{iS}{\hbar}\right) \tag{2.62}$$

We show that the phase is the classical action in the limit that the parameter $\hbar$ is small. We do this by separating out the real and imaginary contributions of the resultant equation. The real part provides the flow of matter and represents its conservation:

$$\partial_k\left(\rho v_k\right) + \frac{\partial \rho}{\partial t} = 0 \tag{2.63}$$

This is precisely the same form we obtained for continuous matter distributions Eq. (2.9). Even in the quantum domain, there is a concept of a continuum distribution that obeys the laws of continuity. In

[12] This and the previous section are advanced. Though their results set the context for our theoretical foundation, they are not needed for engineering calculations of decisions.

making this comparison, the velocity of the electron is defined in terms of the momentum $\mathbf{p}_k = m\mathbf{v}_k = \partial_k S$. The second equation is determined by the imaginary part:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\partial_k S)(\partial_k S) + V = \frac{\hbar^2}{2m}\rho^{-1/2}\partial_k\partial_k\rho^{1/2} \tag{2.64}$$

In the limit that Planck's constant goes to zero, this is the Hamilton Jacobi equation Eq. (2.55) and demonstrates that the phase S is the classical action. It also demonstrates that the paths will be those of least action, whenever the left hand size is significantly greater than the right hand size, which is proportional to $\hbar^2$.

The action described by S corresponds to the classical "rigid" body we might call an electron. The action of the wave however can be associated with the electron viewed as a continuous field defined over space, in the same way we have defined the electromagnetic field. We can determine Schrödinger's equation through a *principle of least action* from the field perspective (Goldstein, 1959):

$$S' = \int \frac{\hbar^2}{8\pi^2 m}\left(\partial_k\psi\partial_k\psi^* + V\psi^*\psi + \frac{\hbar}{2\pi}\left(\psi^*\dot\psi - \psi\dot\psi^*\right)\right)dxdydzdt \tag{2.65}$$

The wave function fields ψ and ψ^* are assumed here to be independent, so the variation equation of least action yield the Schrödinger equation and its complex conjugate. In particular, such equations will yield an energy-momentum stress tensor created from the electron field, and so energy and momentum will flow continuously from one part of space to another. The modern view is to replace the above equation for a non-relativistic field with that for a relativistic field. The current view of physics is that matter consists of quanta of fields as opposed to rigid bodies or particles.

The path integral approach (Feynman & Hibbs, 1965) provides an alternate picture of how the transition is made from the quantum domain to the classical domain in which the *principle of least action* is valid. The path integral approach has been applied to both non-relativistic and relativistic phenomena and so generalizes the Schrödinger approach. The path integral approach provides a unified description of quantum electrodynamics, a fairly complete description that unifies the observational phenomena of quantum mechanics, special relativity and classical mechanics. It provides a conceptual framework for relating the classical action to the phase. We presume that this would be a fruitful approach to use with our *decision process theory*.

Feynman at the outset takes the view that rates are determined by the absolute value of the amplitude (wave function) and that the amplitude is constructed from the action in exactly the same way as the amplitude for light assuming photons are particle-fields: Feynman thus goes back to Newton's original idea that light is composed of particles. Each path contributes a complex number whose phase is the integral of the Lagrangian times the time over the path. The wave function is the sum of these complex numbers.

Since this description is abstract, we provide a simple example (Feynman & Hibbs, 1965) of such an integral for the case of a free particle that we can then compare to Schrödinger's equation for a rigid body (characterized by a mass m constrained so that the motion is described by a single active variable q) constrained to move in one dimension, starting with an expression for the action $S[q]$:

$$S[q] = \int_{t_a}^{t_b} \tfrac{1}{2} m\dot{q}^2 dt \tag{2.66}$$

Feynman's prescription for computing the amplitude for such a constrained particle is to consider all possible paths between the two initial points in coordinate and time $\{q_a \quad t_a\}$ and $\{q_b \quad t_b\}$ and for each path compute the Newtonian action:

$$K(q_b t_b; q_a t_a) = \sum_{allpaths} \exp\left[\frac{i}{\hbar} S[q]\right] \tag{2.67}$$

The expression depends only on the initial and final positions and times. The sum can be carried out by considering a finite number of equally spaced time intervals with the interval of time determined from the time difference and the number of intervals $\Delta t = (t_b - t_a)/(N+1)$, and then taking the limit $N \to \infty$:

$$K(q_b t_b; q_a t_a) = \left(\frac{2\pi i\hbar (t_b - t_a)}{m} \right)^{-1/2} \exp\left[\frac{im(q_b - q_a)^2}{2\hbar (t_b - t_a)} \right] \tag{2.68}$$

By construction this is for times t_b later than t_a: $t_b > t_a$. Because of causality, the amplitude for propagating backward in time is zero. The result obtained in this way is a solution to the Schrödinger equation.

$$\psi(q,t) = K(qt; q_a t_a) = \left(\frac{2\pi i\hbar (t - t_a)}{m} \right)^{-1/2} \exp\left[\frac{im(q - q_a)^2}{2\hbar (t - t_a)} \right] \tag{2.69}$$

Although we start with an idea of rigid bodies, we end with a continuum description or field that describes the electron.

What is helpful about the path integral formulation is that the evaluation of the path simplifies in the limit that the action is very much larger than the Planck constant. The only paths that contribute are the paths near the values where the action is maximal or minimal: the paths of least action. The path integral approach also is insightful when there are two or more classical paths, such as the above problem where the motion is on a ring, *cf.* the review by (Thomas G. H., 1980).

The important thing we learn from this analysis is that there are good reasons to use the *principle of least action* and equally good reasons to take that principle as an approximation. As an approximation, we expect any theory to contain a parameter such as $\hbar$ with dimensions of action. For decision theory, the dimensions would be $[UT]$. What distinguishes our approach from physics is that each decision process has a theory of its own and so has its own quantum of action set by its own $\hbar$. We implicitly acknowledge this by using different units of time characterizing any given type of decision process. Some processes are mere moments, some minutes, days, months or years. What we require in order to apply our *decision process theory* is that we work in the *geometric optics* limit where *actions* are much bigger than $\hbar$.

2.8 ***Outcomes***

We acknowledge that we are not the first to suggest using mechanical models for decisions. We have gained insight from (Von Neumann & Morgenstern, 1944), though there were others before them. The principle of the greatest good for the greatest number was the basis for Bentham utilitarianism (Bentham, 1829). Certainly a principle of greatest happiness resonates with the physical idea of least action: in both cases one imagines an extremum. A physical theory along these lines was proposed by[13] (Edgeworth, 1881) equating the greatest happiness idea to a principle of maximum energy. These approaches lack mechanisms to account for unjust behaviors such as the institution of slavery. They are not set up to allow offsetting effects and so are not complete. It is required of many students in engineering to take a course in ethics in which such issues are discussed in great detail, see for example (Tavani, 2011). The elements missing from economics theories based on principles termed egoistic by their authors are off-setting mechanisms that enforce justice. This is not however, an appeal for a theoretical proof of justice; only a call that a theoretical foundation must allow for such ideas. It is our claim that we extend the more modern discussion of utility by (Von Neumann & Morgenstern, 1944) in a way that allows for such new ideas. There are plenty of mechanisms we can think of that we might want to explore. We return to these issues in chapter 7.

In general terms, in this chapter we have updated the ***decision process theory*** so that it contains not only *non-inertial forces* but *inertial forces.* We have used the *principle of least action* to provide a self-

[13] I am indebted to Mr. K. Kane for bringing this essay to my attention.

consistent view of these effects. Our goal to present a comprehensive and robust framework forced us to include *non-inertial forces* effects due to rotating frames. Decisions involve complex strategies that are executed by ***agents*** or as they are called in game theory, ***players***. These players make choices that in fact may change over time. We have been led to a *decision process theory* in which the strategic choices are represented by *active dimensions* of space, the players provide one or more *inactive strategies* each of which is hidden but manifests as a payoff field associated with that player. The *principle of least action* Eq. (2.31) replaces Eq. (1.13), which is a mechanistic device to identify the equilibrium strategies. By providing different working models for the energy momentum tensor Eq. (2.37) we provide a common and scientific framework to expand our understanding of the decision making process. In the next chapter we will focus on the consequences of players, their isometries and persistent behaviors in this theory.

As in the previous chapter, the attainment of the outcomes in this chapter will be facilitated by doing the exercises in the following section. We list here more detailed outcomes expected by section.

- From section 2.1, the student should understand the need for inertial effects in decision processes and gain an understanding of the form such effects have in a physical based theory.
- From section 2.2, the student will know how to incorporate both inertial media effects as well as non-inertial effects in a unified manner. The prime example of non-inertial effects is that due to the payoff fields of game theory.
- From section 2.3, the student learns how to unify the inertial fields to the payoff fields.
- From section 2.4, the student learns the final unification that leads to a quantitative ***decision process theory***, Eq. (2.31). It is motivated by the need to include *orientation flux fields* in the theory in a unified manner. Measuring distances or time in decision processes is dependent on the frame of reference. The changes are captured in the *orientation flux fields* and related metric fields. Such changes are necessary consequences of the common sense notions that values and time have local meaning, whereas comparing values and time at different points of space and time require some continuous mechanism; that mechanism carries both energy and momentum.
- From section 2.5, the student should be able to use a working model for the energy momentum tensor of the *inertial media*. The student should understand the dynamic equations that result from the conservation laws.
- From sections 2.6 and 2.7, the student should have an appreciation that the *principle of least action* need not be an exact principle. It may fail when the value of the action becomes too small. This is an area for future research, both in terms of observation and theoretical development. It has proven to be an area of great interest in the physical sciences.

2.9 ***Exercises***

1. Give examples of inertial effects from the physical world.
2. Give examples of inertial effects in economics.
3. Show that the ***principle of least action*** Eq. (2.2) leads to the mechanics equations of Lagrange, Eq. (1.9).
4. Show that the application of the *principle of least action* to Eq. (2.17) yields Maxwell's equations, Eq. (1.24).
5. Show that the energy and the momentum components of the electromagnetic field are given by Eq. (2.18).
6. Show that the conservation of the energy momentum tensor for the electromagnetic fields, Eq. (2.19) also leads to Maxwell's equations, Eq. (1.24). What needs to be added to the energy momentum tensor to include the matter contributions?
7. With the help of (Gockeler & Schucker, 1987), show that the *principle of least action* yields Eq. (2.31).
8. Derive the conservation laws Eq. (2.45) from $T^{ab}{}_{;b} = 0$.

9. In a coordinate basis Eq. (1.59), use Eq. (2.34) to demonstrate the following symmetry relationships for the curvature tensor components, which by covariance must hold in any basis:
$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{2.70}$$
10. In a coordinate basis Eq. (1.59), use Eq. (2.34) to demonstrate the following symmetry relationship for the curvature tensor components, which by covariance must hold in any basis:
$$R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab} = 0 \tag{2.71}$$
11. The symmetry relationships can be derived without recourse to the coordinate basis, but from more general principles. Using only the first two (anti) symmetry relations in Eq. (2.70), $R_{abcd} = -R_{bacd} = -R_{abdc}$ and the cyclic symmetry condition Eq. (2.71), which follows from the second Bianchi identity (Gockeler & Schucker, 1987) in a general coordinate system, derive the symmetry relation:
$$R_{abcd} = R_{cdab} \tag{2.72}$$
12. Using the full set of symmetries Eq. (2.70), show that we don't get the cyclic symmetry Eq. (2.71) but the cyclic tensor below is totally antisymmetric in all indices:
$$X_{abcd} = R_{abcd} + R_{acdb} + R_{adbc} \tag{2.73}$$
13. Use the results of the previous exercises to show that the total number of independent curvature components in a space-time of n dimensions is $\tfrac{1}{2}\left(\tfrac{1}{2}n(n-1)\right)\times\left(\tfrac{1}{2}n(n-1)+1\right)$ because of the symmetries Eq. (2.70), less $\dfrac{n!}{4!(n-4)!}$ because of the remaining constraint of setting each independent tensor Eq. (2.73) to zero to satisfy the cyclic symmetry constraint Eq. (2.71). Show that the resultant number of independent curvature components is $\tfrac{1}{12}n^2\left(n^2-1\right)$.
14. Using the average pressure Eq. (2.41) demonstrate the following form for the contracted curvature tensor:
$$R_{ab} = \kappa\left(V_aV_b\left(-(\mu+p)+\frac{\mu-p}{n-1}\right)+p_{ab}-h_{ab}p+h_{ab}\frac{\mu-p}{n-1}\right) \tag{2.74}$$
15. Derive the conservation laws Eq. (2.45) from the form of the energy momentum tensor Eq. (2.43).
16. Show that the conservation of the energy momentum tensor Eq. (2.32) is a direct consequence of the conservation of the Einstein tensor $G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}$, which follows from the definitions of the curvature tensor and its contractions:
$$g^{bc}G_{ab;c} = 0 \tag{2.75}$$