3 Persistent Behaviors

In chapters 1-2, we developed a consistent theory of decisions built from lessons learned from game theory and the physical world. It is a mathematical and engineering language for a ***decision process theory***. Our proposed theory is Eq. (2.31), extended to include time and both active and inactive strategies:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{3.1}$$

It is not *derived* from game theory, physics, biology or chemistry. It is an hypothesis as well as a framework that must be tested by observations. The concepts created in the last two chapters have consequences that are not simply applications of known physical theories, though we may use mathematics and known physical theories to gain insight on how to better our understanding. We are applying the scientific principles to a new physical domain, which may well generate new insights in this domain as well as to existing domains.

The rest of the book is devoted to exploring the consequences of this theory. In the process, we extend previous work (Thomas G. H., 2006). In this chapter, we illuminate ***persistent behavior*** that is distinctive to decision processes, which leads to the notion of ***players*** or ***agents***. For the mathematically oriented reader, we note that persistent behavior as used here corresponds to ***isometry*** transformations that leave the value of the metric-fields invariant. Each *isometry* requires the existence of a vector field with special properties. The collection of all *isometries* forms a mathematical Lie Algebra, which characterizes the ***internal symmetry*** of the theory. Because of this, a player's or agent's *persistent behavior* defines a well-defined and persistent mathematical structure.

We explore the consequences of the theory by examining it in several frames of reference or bases. We look at the theory first in the ***normal-form coordinate basis***. In this basis the persistent behaviors are characterized by the property that all metric and orientation potentials are independent of the set of internal player dimensions. The consequences of persistency in this basis manifest in the field equations, which are examined in this chapter in a specific gauge, the *harmonic gauge*. We look at the covariant expression of persistency and define *isometry*. We gain additional insight into the consequences of persistency by looking at the behaviors in a ***co-moving coordinate basis***, which in addition gives us a potentially simpler numerical approach to the field equations that we follow in later chapters. Our initial focus will be on an ***orthonormal co-moving coordinate basis***. A focus in later chapters will be on a ***holonomic co-moving coordinate basis***; in this chapter we define what we mean by ***holonomic***. Finally we introduce the ***player fixed frame model*** that provides a class of models that highlights the ***vorticity*** aspects of the theory and may be solved using known current numerical techniques. We use such numerical results in later chapters to illustrate how our *decision process theory* provides a common ground for discussion of economic issues. The model allows us to extend, using an electrical engineering analogy, behaviors that are like DC circuits, to behaviors that are like AC circuits.

3.1 *Coordinate and non-coordinate bases and potentials*

In the physical world, we take the ability to measure distances and intervals of time for granted. This assumption is fundamental to our approach to the physical sciences. We assume that we can determine a physical location and a time for each element of a physical event. Because of that, we can assign a distance between physical events. For these and other reasons, we feel confident that we can describe the dynamic behaviors of physical processes. We can describe the events as they evolve in time.

In this book we assert that we have the same ability with decision processes. Though we have not done that here, we suggest that this assumption is one that needs to be established empirically; we have looked at some data and indeed do observe that it appears to be provisionally true.

We adopt the ***continuity viewpoint*** from the physical sciences that for each decision event, at each moment of time, there is a local region in which each strategic choice has a value that varies only slightly from the event in question and that furthermore, the values change only slightly at nearby times. We make this mathematically precise by asserting that there always exists a ***coordinate basis*** at each point of strategic space time: there is a coordinate value (for time and each strategic choice) that has the same value not only at that point but on an extended surface that contains that point. A complete set of independent surfaces through the point then provides a coordinate basis.

The coordinate value at a point corresponds to a surface through that point. Coordinates of this type are called ***holonomic*** [Cf. section 1.7][14]: the coordinates can each be derived from a potential and the gradients along each coordinate direction, the coordinate vectors ∂_μ, mutually commute, $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$. Such *holonomic* coordinates implement our belief of how distances are measured between decision process events in time. Thus we use coordinates here in the same way electrical engineers use the word potential fields.

Prior to Einstein, our common sense notion was that we could get by with a single and global coordinate basis: a universal view from which to view interactions. This common sense notion we can think of as Newtonian: it corresponds to a frame of reference in which space time is essentially flat. Einstein suggested that this view is too simplistic. Consider a well know object, the sphere. A map maker takes a patch of a sphere and maps it onto a sheet of paper, the chart. This mapping creates the coordinates (local latitude and longitude) that are *locally holonomic*. However, as is well known, the sphere can't be mapped onto a single chart; at least one point will be left out. More than one chart is needed because of the left out points. Our common sense view must be extended to describe even such simple and well-known surfaces. Our use of coordinate bases will be local therefore, not global. Multiple charts will in general be needed to describe even rather simple structures and thus we allow for multiple charts in *decision process theory*.

For coordinates that are *locally holonomic*, we expect certain mathematical properties. Each constant surface that corresponds to a holonomic coordinate has the attributes of a potential field. Potential fields generate a unique vector at each point on the surface that is normal to the surface. In general, parallel surfaces generate the same vector fields. To gain more insight in potential fields, you may recall potentials φ whose gradient $\partial_a \varphi$ determines the properties of the vector field that is normal to the surface of constant potential. Surfaces of constant potential are used in engineering and physics. For example the potential surfaces within a capacitor of arbitrary shape determine the electric field. The gravitational potential in Newtonian mechanics defines surfaces that map the gravitational forces. The potentials provide a convenient method to discuss physical effects associated with vector fields. The vector fields are typically those that more directly determine dynamics, not the potentials. Moreover, not every vector field that determines physical behaviors can be written as the gradient of a potential.

Although locally holonomic coordinate bases provide an intuitive meaning for measurements, there are also advantages of looking at global coordinate systems that are not holonomic. They may better illuminate curvature effects. As a simple example, recall that we live on a rotating sphere whose holonomic coordinate system would be fixed. However, a rotating coordinate system has advantages because it corresponds to the world we actually see. In many ways the rotation of the earth is *hidden* from us. We must then capture the effects of our *hidden* rotation as two separate effects: ***Coriolis effect*** and ***centripetal acceleration***.

Similarly, we suggest that the concept of ***persistency*** that underlies the existence of players and agents is described on the one hand by the locally holonomic *hidden dimensions* of section 1.9 and on the

[14] The idea in physics is that simple systems are integrable. More complex systems are ones with constraints, but if the constraint equations are each integrable, the remaining variables also be integrable. For decisions we envision that the active and inactive variables are those that remain after all the constraints have been imposed. The assumption is that the remaining variables will be integrable and correspond to variables that are exact as defined in the following paragraphs.

other hand is described by a fixed non-holonomic frame. We illuminate some of these ***persistent*** attributes in the ***normal-form coordinate basis***, which we introduced in section 1.9. Using the frame independent *principle of least action*, which provides *decision process theory* Eq. (3.1), dynamics can be described by the ways in which these persistent attributes interact.

To include frames that don't form a *holonomic coordinate basis*, we refine our mathematical distinctions. We start with the basic idea that in a *holonomic coordinate basis*, each vector field can be derived from a potential field. We recall that a vector field $\mathbf{U}$ when thought of as a 1-form, Eq. (1.50) discussed in section 1.7, has a potential only if it is ***exact***, $d\mathbf{U}=0$. In a *holonomic frame*, this reduces to the requirement that $\partial_\nu U_\mu = \partial_\mu U_\nu$, which is the necessary and sufficient condition for there to exist a potential φ such that $U_\mu = \partial_\mu \varphi$. A 1-form is *exact* when we can write it in terms of this potential: $\mathbf{U}=d\varphi$.

We write these as covariant requirements. Starting from a holonomic set of frames dx^μ, we construct a new set of (potentially non-holonomic) frames $E^\alpha = E_\mu{}^\alpha dx^\mu$ using the *gauge transformation* $E_\mu{}^\alpha$. An ***exact*** 1-form $d\mathbf{U}=0$, will be transformed to coordinates:

$$U_\alpha = E^\mu{}_\alpha \partial_\mu \varphi \equiv \Delta_\alpha \varphi \tag{3.2}$$

The requirement $\partial_\mu U_\nu = \partial_\nu U_\mu \Leftrightarrow U_{\nu;\mu} = U_{\mu;\nu}$ in the old frame, takes the covariant form in the new frame:

$$U_{\alpha;\beta} = U_{\beta;\alpha} \tag{3.3}$$

Though the components are determined by the potential using a differential operator, these differential operators in general don't commute, $\Delta_\alpha \Delta_\beta \neq \Delta_\beta \Delta_\alpha$.

A second important distinction for *holonomic coordinate systems* is the commutativity of the differential operators. In differential geometry, any vector field X^μ can be associated with a differential operator $\Delta_X \equiv X^\mu \partial_\mu$. The commutation between two such operators is defined to be a new covariant operator, called the ***Lie product*** of the vectors:

$$\Delta_{\mathbf{X}}\Delta_{\mathbf{Y}} - \Delta_{\mathbf{Y}}\Delta_{\mathbf{X}} = \left(X^\nu Y^\mu{}_{;\nu} - Y^\nu X^\mu{}_{;\nu} \right)\partial_\nu \equiv \left[\mathbf{X},\mathbf{Y}\right]^\mu \partial_\mu = \Delta_{[\mathbf{X},\mathbf{Y}]} \tag{3.4}$$

The covariant requirement that a frame be holonomic is that the 1-forms are all exact and the Lie product of any pair of frame vectors is zero. Conversely, for frames that are not holonomic we expect at least one of these properties to be absent.

The advantage of the covariant expressions is that we can investigate specific attributes of a *holonomic frame* that are present or missing in other bases, such as the *normal-form coordinate basis*. So for example using the definition of exactness, Eq. (3.2), with Eq. (1.74), which provides the orientation potentials used in the covariant derivatives, the assumption that the transformations are functions only of the active strategies leads to the determination that the active coordinates remain exact (as they are not transformed) but the (transformed) inactive coordinates need not be exact. Further, it is not difficult to show (see exercises 1-3 at the end of the chapter) that the Lie products of the transformed inactive strategies are zero and that the Lie products of the transformed active strategies are effectively zero.

3.2 *Normal-form coordinate basis—hidden symmetries*

You might think that the most convenient frame of reference would always be a coordinate or holonomic basis. Such bases lead to differential equations that have been well studied and provide necessary information about existence and properties of solutions. However, such local bases don't necessarily provide the best global view, especially if there are symmetries. For example, in *decision process theory*, persistency determines the property of the distance measure, Eq. (1.68):

$$ds^2 = \gamma_{jk}\left(d\xi^j + A_a^j dx^a\right)\left(d\xi^k + A_b^k dx^b\right) + g_{ab}dx^a dx^b \tag{3.5}$$

The symmetry is that the metric elements are independent of the inactive coordinate vectors ξ^j. We say that these coordinates are ***hidden***. If the expression is expanded, the most general form of the distance measure is recovered without changing this "hidden" property. Writing the measure of distance as Eq. (3.5) emphasizes that many equivalent choices of variables that transform only the hidden coordinates will lead to the same end result. Distances are made up of two orthogonal contributions: the first an inactive or internal contribution and the second an active contribution. All results in *decision process theory*, such as the above measure of distance, are independent of the choice of gauge. In particular, a subset of these gauge transformations that leaves the inactive variables *hidden* must leave all results of the theory unchanged. We define these transformations in more detail.

The distance measure is expressed in terms of locally holonomic coordinates $x^\mu = \{\xi^j \quad x^a\}$ that represent the inactive and active strategies respectively (with time treated here as active). These coordinates are not in general orthogonal. The inactive coordinates ξ^j are *hidden* in the sense that the tensors $\{\gamma_{jk} \quad A^j{}_a \quad g_{ab}\}$ are independent of these strategies. We can verify that the following gauge transformation $\xi^j \to \xi^j - \Lambda^j, \quad A^j{}_a \to A^j{}_a + \partial_a \Lambda^j$ leaves the inactive strategies *hidden*. For example, the distance Eq. (3.5) does not change under these transformations as long as the inactive and active metric elements $\gamma_{jk} \quad g_{ab}$ don't change (are gauge invariant). Calculations done in the coordinate basis typically involves a significant amount of extra algebra because many of the intermediate results are gauge dependent. For example, these intermediate results may depend explicitly on the vector potential A^j_a as opposed to the gauge independent payoff field $F^j{}_{ab}$, the active metric g_{ab} or the inactive metric γ_{jk}.

Gauge independence for *decision process theory* follows because the *principle of least action* is frame independent. This covers frames that are *locally holonomic* or frames that are not. Therefore without losing any generality, by picking a non-holonomic frame, it may be possible to reduce significantly the algebra necessary to obtain the desired gauge independent results. We illustrate this by looking at our theoretical framework in the ***normal-form coordinate basis*** using the gauge transformation formalism from section 1.7 and Eq. (1.69) in section 1.9:

$$
\begin{aligned}
\mathbf{U}^a &= dx^a \\
\mathbf{U}^j &= d\xi^j + A^j_a dx^a
\end{aligned}
\tag{3.6}
$$

This basis, which is suggested by the invariant distance Eq. (3.5) provides a natural gauge invariant and orthogonal split into active coordinates, which are holonomic (on the active subspace only, see exercise 3) and inactive coordinates that are gauge invariant but not holonomic.

We adopt the following conventions. Unless specifically noted, we label the active ***exact*** dimensions, which are time and the active strategies, by the indices at the beginning of the alphabet $\{a,b,c,\cdots\}$. We label the inactive strategies, which are not *exact* in this basis, by the indices later in the alphabet $\{i,j,k,\cdots\}$. We use the following notations for the determinant of the inactive metric components γ_{jk} and active metric components g_{ab}:

$$
\begin{aligned}
\gamma &= \det \gamma_{jk} \\
g &= \det g_{ab}
\end{aligned}
\tag{3.7}
$$

Noting that these two sets are orthogonal to each other, the determinant of the full metric for the space is the product γg, which can be either positive or negative. We indicate the absolute value of a quantity in the usual way, for example $|\gamma|$ is the absolute value or magnitude of the inactive determinant.

In the *normal-form coordinate basis*, the payoff matrix is determined by the differential of Eq. (3.6) that is manifestly covariant:

$$d\mathbf{U}^j = \mathbf{F}^j = \tfrac{1}{2} F^j{}_{ab}\mathbf{U}^a \wedge \mathbf{U}^b$$
$$F^j{}_{ab} = A^j_{b;a} - A^j_{a;b} \qquad (3.8)$$

In the *normal-form coordinate basis*, this expression is gauge invariant:

$$F^j{}_{ab} = \partial_a A^j_b - \partial_b A^j_a \qquad (3.9)$$

The expression Eq. (3.8) is derived from Eq. (2.71).

We thus have a basis in which the calculations of gauge invariant results are obtained directly, albeit in a frame that is non-holonomic. The degree to which the basis is non-holonomic is specified by the payoff field. As a notational aside, we make frequent use of raising and lowering the indices using the metric, so for example we can write:

$$F_j^{ab} = \gamma_{jk} g^{ac} g^{bd} F^k_{cd} \qquad (3.10)$$

This highlights that there are no non-zero metric components that mix active and inactive strategies in the *normal-form coordinate basis*. As a further physical aside, we note that the *Coriolis effect* and *centripetal acceleration* on the earth in which the inactive coordinate is time, respectively would be the "payoff" tensor for the ***Coriolis effect*** and the gradient of the inactive metric for the ***centripetal acceleration***, identified as gravitation.

3.3 ***Normal-form coordinate basis—harmonic gauge***

Because the *principle of least action* remains unchanged under local linear transformations, *i.e.* gauge transformations, the resultant field equations have a corresponding number of degrees of freedom that are larger than the physically distinguishable number: many choices of frames solve the same set of equations. By going to the *normal-form coordinate basis*, we have reduced, though not entirely eliminated those degrees of freedom. What are physically meaningful are the set of coordinate *vectors* and the gauge-choice that removes all unfixed degrees of freedom. This freedom of choice is not without precedent in physics: in electrical engineering for example, there are many choices of potentials Eq. (1.21) that lead to the same electric and magnetic fields. In *decision process theory*, we are also free to make such gauge choices, which lead to identical outcomes for decision processes.

What is unusual about this is that it goes against the common Newtonian notion that there is a unique global space and time not subject to arbitrary mathematical gauge choices. The Newtonian view is appropriate to flat space not curved space. For this reason, we go against this view and allow gauge arbitrariness in the choice of pure strategies and time, which allows for dynamics that reflect a topological and geometric structure that will be constrained by its success in matching observational data. The gauge freedom is analogous to the freedom of using "charts" for navigation: each chart treats the earth as being flat or Euclidean. Because they describe an object that is round, they don't exactly match up. The mapping that matches them is the gauge transformation. The gauge freedom is that at any point, we are free but not required to consider the space (our chart) to be flat. Though there is gauge freedom, it is not a freedom that can be removed as an attribute of the theory: topological objects with curvature such as the sphere require multiple charts. We assert that decision processes generates curvature and require theories with this gauge property.

To solve the local field equations, we can pick any gauge. The solution will be valid in some region around our initial conditions. Our choice is the ***harmonic gauge*** described in the literature, (Wald, 1984). We adapt an argument from that literature to apply ***harmonic coordinates*** to the ***normal-form coordinate basis***. We start with the field equations Eq. (3.1) for *decision process theory* that is the consequence of the *principle of least action*. We write these equations in the *normal-form coordinate basis.* The argument assumes we have a set of coordinates, x^a, which are scalar functions of y^a in another basis in which the metric $\bar{g}^{ab}(y)$ is known in terms of these coordinates y^a.

We define the active coordinates x^a in the *normal-form coordinate basis* using the following differential equation, where the Greek indices $\{\mu,\nu\}$ span both the active and inactive dimensions of the *normal-form coordinates*:

$$\bar{g}^{\mu\nu} x^a{}_{;\mu\nu} = 0 \tag{3.11}$$

This differential equation can be written in any frame, so in particular can be written in the normal-form coordinate basis with metric g^{ab}. The differential equations can be expanded using Eq. (1.74) and written in terms of the determinants Eq. (3.7) and the active partial derivatives:

$$\frac{1}{\sqrt{|g\gamma|}}\partial_b\left(\sqrt{|g\gamma|}g^{bc}\partial_c x^a\right) = 0 \tag{3.12}$$

These differential equations and suitable initial conditions have a unique solution, which provides a solution for the normal-form coordinates. In the normal-form coordinate basis, the differential equation can also be written in a form that puts constraints on the number of independent metric potentials g^{ab}:

$$\frac{1}{\sqrt{|g\gamma|}}\partial_b\left(\sqrt{|g\gamma|}g^{bc}\partial_c x^a\right) = 0 \Rightarrow \frac{1}{\sqrt{|g\gamma|}}\partial_b\left(\sqrt{|g\gamma|}g^{ba}\right) = 0 \tag{3.13}$$

The coordinates x^a that satisfy this condition are the ***harmonic coordinates*** and Eq. (3.13) is the ***harmonic gauge condition***. Without excess degrees of freedom, these conditions, the initial conditions and the field equations completely specify the solution to the field equations, which we turn to next.

3.4 *Normal-form coordinate basis—field equations*

In this section we develop the field equations with sufficient information that the interested student can verify the results. The important steps needed to obtain the results are specified as exercises at the end of this chapter, so only the approach, attributes and interpretation will be given in this section.

The field equations in the *normal-form coordinate basis* are obtained directly from the curvature tensor, exercise 6, Eq. (3.77), exercise 7, Eq. (3.78) and exercise 8, Eq. (3.79). Because we operate in the *normal-form coordinate basis*, the gauge invariant results, relative to the hidden dimensions, are obtained directly. Given the full curvature tensor components, the field equations Eq. (3.1) are obtained by contracting the curvature tensor with the metric. The number of equations depends on the ***space dimension*** n, which is the total number of active and inactive dimensions. The resultant equations are of three basic types, depending on whether the indices are all active, all inactive or mixed. Technically the field equations are restricted to fields that are functions only of the active variables, so on this subspace the active variables x^a are holonomic and ordinary rules of calculus apply. With the choice of *harmonic gauge* these equations along with suitable boundary conditions have unique solutions. We now discuss the equations that result for these three cases.

We start with the field equations for the payoff tensor, which result from the full set of field equations with mixed indices:

$$\tfrac{1}{2} g^{bd}\partial_b\left(g^{ac}\gamma_{jk}F^k{}_{cd}\right) = \kappa T_j^{\ a} \tag{3.14}$$

This is the covariant form in *harmonic coordinates*, and generalizes Maxwell's Eq. (1.24). There is one such equation for each player or agent in the decision process. There is a source contribution $T_j^{\ a}$, which may have behaviors that are common to all players. There are contributing sources as well from strategy and time behaviors of the active and inactive metrics: such behaviors are *frame rotation* effects that may be consequences of the particular observation frame. These effects show up more clearly if we solve the equations using vector potentials. In this case the vector potentials and their gauge dependence provide a useful way to simplify the equations and interpret the results.

We use the potential form Eq. (3.8) along with the following gauge conditions for the potentials, which are a type of *harmonic gauge* applied to the inactive space (Thomas G. H., 2006):

$$g^{ab}\partial_b A_a^j = 0 \qquad (3.15)$$

Using these gauge choices, we write the generalized Maxwell's equations for the vector potentials for each player:

$$\tfrac{1}{2} g^{bd}\partial_b\partial_d A_a^j + \tfrac{1}{2}\partial_a g^{bd}\partial_b A_d^j + \tfrac{1}{2}\partial_b g_{ae} g^{ec} g^{bd} F^j{}_{cd} - \tfrac{1}{2} g^{bd}\gamma^{jl}\partial_b\gamma_{lk}F^k{}_{ad} = -\kappa T_a^j \qquad (3.16)$$

The similarities and differences to the physical equation for electrodynamics can be expanded upon. The similarity is that for each agent, we have a wave equation, so we expect signals to propagate independent of any matter medium with a common velocity. Each wave equation generalizes the wave equation for electromagnetic fields Eq. (1.16).

The substantial differences are as follows. There are multiple wave equations: there is one equation for each player. Each player feels the inertial forces $T^j{}_a$, which are similar for each player for simple models, such as the perfect fluid model, Eq. (2.37) and therefore the vector potential solutions for each player will be forced to be similar. As a special case, this may generate the Nash equilibrium in game theory, which postulates that equilibrium exists when all players subscribe to the same rules. More generally, we see how these same rules enforce common behaviors.

Another set of differences is the existence of "currents" that result not from external sources, but purely from *frame rotation* effects, gradients of the active or inactive metric. We see a contribution corresponding to each. There is mixing between players, a phenomenon that is special to this theory of decisions. The last term of Eq. (3.16) on the left hand side mixes the roles of the players. In order for this to be possible, the scalar fields must not be constant. The scalar fields mix internal and active geometry effects.

The field equations that correspond to the active space depend on the persistent attributes of the players:

$$\bar{R}_{ab} - \tfrac{1}{2} g_{ab}\bar{R} = -\kappa T_{ab} - \kappa T_{ab}^{payoff} - \kappa T_{ab}^{inactive} \qquad (3.17)$$

We see three distinct contributions to the "bar" or active curvature components on the left-hand-side of the equation that are computed in the active subspace. In other words, the reduced tensor $\bar{R}_{ab}$ is given directly by Eq. (2.36), which includes partial derivatives through the second order in the active metric. Its form will be simplified further by the choice of the *harmonic gauge*. Though the resultant equations are difficult to solve, we can say a few things based on general principles.

The first contribution on the right-hand-side of Eq. (3.17) is from the matter field source $-\kappa T_{ab}$. As in physical theories, for a time isometry (defined in the next section) and very weak couplings κ, the metric is well approximated by the Minkowski metric with the exception of the time component of the metric $g_{00} \cong 1-\Phi$. The deviation from unity is a static (time independent) field that is determined by the matter density:

$$\nabla^2\Phi = \kappa\rho \qquad (3.18)$$

This is Poisson's equation and reflects that matter attracts other matter. In the dimensions of 3+1 space-time, this is an inverse square attraction. In a strategic space with n dimensions this force law is generalized.

More generally, we can see that if time is an isometry as defined in the next section, there will be two effects corresponding to the Coriolis effect and centripetal acceleration. In general relativity the "***centripetal acceleration***" is determined by the time component of the metric and the corresponding "***Coriolis effect***" is ***gravitomagnetism*** (Ryder, 2009, p. 180ff) and can be measured with gyroscopes and clocks by means of satellites. It is interesting that there is a connection between payoffs, magnetism and Coriolis. The source of the connection is the underlying symmetry or hidden variable. To make the connection complete for electromagnetism, we use (Kaluza, 1921) and (Klein, 1956) for the Kaluza-Klein theory and their proposed hidden fifth dimension.

It is worth emphasizing that these effects do not arise from a physics analogy but from the mathematical structure of the proposed *decision process theory* and general principles of differential geometry. They all share the same underlying mathematics. We find it significant that the gradients of the orientation potentials provide a basis for an attraction that exists independently of the payoffs. In whatever frame we work, an important effect is that of acceleration. This is the symptom of the force of attraction. It is also the cause, at the global level, of the non-existence of a global frame that is simultaneously holonomic.

The matter density is not the only contribution to the energy and momentum of the active space. There is also the effect from the payoff fields:

$$2\kappa T_{ab}^{payoff} = \gamma_{jk} F_{ac}^{j} F_{bd}^{k} g^{cd} - \tfrac{1}{4} g_{ab} F_{k}^{cd} F_{cd}^{k} \tag{3.19}$$

To physicists and electrical engineers, this is clearly an analogous form of the energy momentum fields of Maxwell, Eq. (2.18) applied to the case of multiple payoff fields corresponding to the players involved in a decision. As pointed out by Einstein, light has inertia. What we say in *decision process theory* is that payoffs have inertia.

Again, these arise not from analogy but from the *decision process theory*. These are specific consequences of the existence of persistent behaviors. Einstein was the first to suggest that such radiation fields carry real inertial effects just like matter. He predicted that light would bend as it passes close to a star as a consequence of this inertial effect. Here also there will be real inertial effects that follow from the presence of the payoff fields. Qualitatively, it says that transactions with large payoff matrix elements carry more energy and will have more impact on dynamics than transactions with small payoff matrix elements. This is different from game theory in that two games can have proportional payoff matrices implying the same strategic consequence, even though one game might have small matrix values and the other large.

The last contribution to the active space field equations Eq. (3.17) comes from the inactive metric:

$$\kappa T_{ab}^{inactive} = \begin{pmatrix} \tfrac{1}{2}\gamma^{jk}\gamma_{jk;ab} + \tfrac{1}{4}\partial_a \gamma^{jk} \partial_b \gamma_{jk} \\ -\tfrac{1}{2} g_{ab} \left(\gamma^{jk} g^{cd} \gamma_{jk;cd} + \tfrac{1}{4} g^{cd} \gamma^{mn} \gamma^{jk} \gamma_{mn;c} \gamma_{jk;d} + \tfrac{3}{4} g^{cd} \gamma^{jk}{}_{;c} \gamma_{jk;d} \right) \end{pmatrix} \tag{3.20}$$

The inactive metric behaves as a matter field with a specific form determined by the field equations. The inactive metric field also carries inertia. It can be bent by strong gravitational sources. The existence of this field was not foreseen by our initial analysis in section 1.4. It is not uncommon however in theories based on those of (Kaluza, 1921) and (Klein, 1956).

The last set of field equations obeys massless wave equations:

$$\tfrac{1}{2} g^{ab} \partial_a \left(\gamma^{jl} \partial_b \gamma_{lk} \right) = \tfrac{1}{4} F^{j}{}_{ab} F_{k}^{ab} - \kappa T^{j}{}_{k} \tag{3.21}$$

In the absence of sources, this is a massless wave equation (a Klein-Gordon field in physics texts). These fields (along with other possible fields) in Einstein's theory could describe gravitational waves. Thus even in empty space there may be influences that propagate with the speed of light.

In general, the active and inactive field equations determine metric potentials that are not Minkowski. Thus there is no *a priori* justification for requiring the active and inactive metrics to be flat. Even in spaces with no inertial sources, the payoffs contribute as sources and provide the source for curvature Thus given the equations, the next logical step is to apply the ***decision process theory*** using these equations for decision problems, solve the equations for such problems (numerically), analyze the results and based on the results, refine the theory. In the process we expect to grow a quantitative understanding of the decision process.

With suitable computing power, we have sufficient information to apply the *decision process theory* to a wide variety of problems. As anyone in engineering knows however, it is not just the theoretical ability to solve problems that is of singular importance, it is also the practical ability to find sets of problems that can be solved and from them, the knowledge that gives the ability to build structures (bridges, circuits, etc.) that do desired things.

We hope to demonstrate such a set of solvable and useful problems exist, in analogy to the set of problems that have been found in electrical engineering. In that discipline, great strides are made using the insights gained from electrostatics and magneto-statics, coupled with introducing time dependence using ***phasors*** (in mathematical terms, using Fourier series). The same mathematics of static solutions can then be applied to phasor solutions to gain insight into dynamic steady-state behaviors. We are able to carry out this program for a class of models (chapter 4). It is our current belief that the local equations in the normal-form coordinate basis don't always give a clear large scale global picture; but insight might be gained in other frames. In the next section, we return to the notion of persistency and express it in more general terms so that we can consider the field equations in other frames, specifically we will look at the equations in the orthonormal co-moving reference frames in section 3.6.

3.5 *Isometry and hidden symmetry*

Our intuitive definition of persistency (isometry) required the existence of the *normal-form coordinate basis*, in which the metric is independent of the inactive dimensions: these dimensions are *hidden* reflecting a hidden symmetry. Translations along each of these inactive dimensions leave the metric unchanged. In this section, we formulate this hidden symmetry covariantly so that we can transform the concept to other frames.

In differential geometry, a transformation that leaves the metric elements unchanged is called an ***isometry***. A necessary and sufficient covariant condition for there to be an *isometry* is that there is a vector field K_μ satisfying the "Killing" conditions:

$$K_{\mu;\nu} + K_{\nu;\mu} = 0 \qquad (3.22).$$

Since this is a covariant relation, to prove this we choose a holonomic coordinate system to which we can always transform the coordinates so that the vector field is unity along the dimension ξ and zero along the other coordinates. The above condition implies that

$$\begin{aligned}
&K_\mu = g_{\mu\nu} K^\nu = g_{\mu\xi} \\
&K_{\mu;\nu} = \partial_\nu g_{\mu\xi} - \omega_{\xi\nu\mu} \\
&K_{\mu;\nu} + K_{\nu;\mu} = \partial_\nu g_{\mu\xi} + \partial_\mu g_{\nu\xi} - 2\omega_{\xi\nu\mu} = \partial_\nu g_{\mu\xi} + \partial_\mu g_{\nu\xi} - \left(-\partial_\xi g_{\mu\nu} + \partial_\mu g_{\nu\xi} + \partial_\nu g_{\xi\mu}\right) \\
&K_{\mu;\nu} + K_{\nu;\mu} = \partial_\xi g_{\mu\nu} = 0
\end{aligned} \qquad (3.23)$$

Therefore given the condition Eq. (3.22), there is a frame in which the metric is independent of the dimension ξ.

Conversely if there is a frame in which the metric is independent of the dimension ξ, we pick K^μ to be the vector field that is unity along that direction and zero otherwise. The above argument again holds and we deduce Eq. (3.22). Since it is a covariant relation that holds in one frame, it therefore holds in all frames. This relationship is attributed to Killing.

If there are two isometries there will be two Killing vectors K^μ and L^μ. From two vector fields we create a new vector field called their ***commutator*** or ***Lie product*** Eq. (3.4):

$$\left[\mathbf{K},\mathbf{L}\right]_\mu = K^\nu L_{\mu;\nu} - L^\nu K_{\mu;\nu} \qquad (3.24)$$

We leave as an exercise that the commutator of two Killing vector fields is itself a Killing vector:

$$\left[\mathbf{K},\mathbf{L}\right]_{\mu;\nu} + \left[\mathbf{K},\mathbf{L}\right]_{\nu;\mu} = 0 \qquad (3.25)$$

This has far reaching consequences. The set of all Killing vectors, which includes their commutators, form a Lie Algebra, with the *Lie product*. The isometries of the theory therefore generate a local symmetry group (*i.e.* a symmetry group at each point in space), which has the same structure at every point. This provides substance to the notion that isometries are ***persistent*** and reflect the local symmetry group.

Isometries reflect properties of what we term an ***internal group***, which gives substance to invariance and *persistency* associated with the *inactive strategies*. In particular, we see that the concept of ***independent players*** has in fact been framed as a group theoretical statement that there is a commutative subgroup at every point with exactly the same group structure. Because the operators commute we can always find a frame in which the metric components are simultaneously independent of all the associated inactive dimensions. We now have a firm theoretical foundation. We next investigate the consequence of these *isometries* in a specific frame, a co-moving frame, using the *harmonic gauge*.

3.6 *Co-moving orthonormal coordinate basis*

Field equations of the type Eq. (3.1) and their solutions have been extensively studied, for example (Wald, 1984, p. 252ff), and (Hawking & Ellis, 1973). The essential point for us is that solutions to these field equations exist and have reasonable properties if the metric, orientation potentials, matter fields and gradients of these are specified initially on a surface "orthogonal to time". The field equations then determine the evolution of the metric, orientation potentials, matter fields and flows on all surfaces that are "later". Our goal is to use these solutions to *decision process theory* to better understand the decision process. We believe this understanding must be based on a *quantitative* understanding of the consequences of these field equations. To this end it is natural for us to adopt and adapt the approach to solving equations of this type that has proven useful in the past (*Cf.* section 3.3).

For these reasons we look at the equations in a frame in which one direction is along the energy flow vector that arises naturally in considering the inertial fields Eq. (2.37). The energy flow direction is a natural "time" direction. In this frame we are co-moving so the vector has only a time component and the remaining directions are orthogonal unit vectors with respect to the metric. We further look for a system in which this metric is globally Minkowski. Because the metric is constant, in general the coordinates will not be holonomic. We call this frame the ***co-moving orthonormal coordinate basis*** or if there is no ambiguity, the ***co-moving basis***.

The Minkowski metric we denote as $m_{\alpha\beta}$. It is diagonal with a value $+1$ for the time like indices and -1 otherwise. For our discussion in this section[15], we use Greek letters in the beginning of the alphabet $\{\alpha,\beta,\gamma,\cdots\}$ to represent the dimensions in the co-moving basis and the Greek letters in the middle $\{\mu,\nu,\lambda,\cdots\}$ to represent the dimensions in the *normal-form coordinate basis* when we don't want to specify whether the dimensions are active or inactive. Thus in this latter case the indices span both active and inactive dimensions.

In each frame there will be orientation potentials, which transform not as tensors but as specified in the frame transformation Eq. (1.63). In each frame the covariant derivative depends on these orientation potentials. We use the notation $X^{\mu}{}_{;\nu}$ for example, to indicate the covariant derivative that depends on the potentials $\varpi^{\mu}{}_{\nu\lambda}$ in the *normal-form coordinate basis*, Eq. (1.52). Ordinarily, the notation would be adequate for any frame, however when we switch back and forth between two frames, it is helpful and less ambiguous to have a special notation for the covariant derivative in the co-moving frame as well as a notation for the potentials in that frame. We use $X^{\alpha}{}_{|\beta}$ for the covariant derivative and $\omega^{\alpha}{}_{\beta\gamma}$ for the orientation potentials in that frame. To carry out our program of solving the field equations, which are frame covariant, we use the transformation properties of these potentials based on their gauge properties Eq. (1.63).

We obtain the transformation properties of the orientation potentials from the transformation properties of the vector $\bar{X}_{\mu}$ in the *normal-form coordinate basis* and its covariant derivative. The transformation is determined by the gauge transformation matrix $E^{\mu}{}_{\alpha}$ that takes the vector in the *normal-form coordinate basis* to the co-moving frame:

[15] Later, we introduce the player fixed frame model in which we make further distinctions between some of these orthonormal coordinates, which we associate with "proper" active strategies and some we associate with "proper" inactive strategies (players).

$$\begin{aligned} X_\alpha &= E^\mu{}_\alpha \bar{X}_\mu \\ X^\alpha &= E_\mu{}^\alpha \bar{X}^\mu \\ E_\mu{}^\alpha E^\nu{}_\alpha &= \delta^\nu_\mu \\ E^\mu{}_\alpha E_\mu{}^\beta &= \delta^\beta_\alpha \end{aligned} \tag{3.26}$$

We also provide above the inverse transformation. We raise and lower indices using the metric in the appropriate basis:

$$\begin{aligned} E_{\mu\alpha} m^{\alpha\beta} E_{\nu\beta} &= g_{\mu\nu} \\ E_{\mu\alpha} g^{\mu\nu} E_{\nu\beta} &= m_{\alpha\beta} \end{aligned} \tag{3.27}$$

The content of the first of these equations is that if the frame transformations $E_{\mu\alpha}$ are known in the co-moving orthonormal coordinate basis, then the *normal-form coordinate basis* metric components are determined. Moreover, since the normal-form metric has the property that $g_{aj} = 0$ for all mixed tensors between the active and inactive space, the frame transformation will have to enforce this property.

The frame transformations determine not only the metric elements from Eq. (3.27), but the payoff matrix Eq. (1.73) using the defining equations for the 1-forms:

$$d\mathbf{U^j} = \tfrac{1}{2} F^j{}_{ab} \mathbf{U^a} \wedge \mathbf{U^b} \tag{3.28}$$

The differential 2-form $d\mathbf{U^j}$ can be computed from the transformation from $\mathbf{E}^\alpha$ and the corresponding 2-form based on zero torsion Eq. (1.54):

$$\begin{aligned} \mathbf{U^j} &= E^j{}_\alpha \mathbf{E}^\alpha \\ d\mathbf{U^j} &= -E^j{}_{\alpha\backslash\beta} \mathbf{E}^\alpha \wedge \mathbf{E}^\beta \end{aligned} \tag{3.29}$$

Comparing the two expressions we find that each player field $\mathbf{E^j}$ acts as the vector potential for the payoff in the co-moving basis:

$$F^j{}_{ab} E^a{}_\alpha E^b{}_\beta = E^j{}_{\beta\backslash\alpha} - E^j{}_{\alpha\backslash\beta} \equiv f^j{}_{\alpha\beta} \tag{3.30}$$

Thus whenever the transformations are determined in the co-moving basis, the ***co-moving payoffs*** $f^j{}_{\alpha\beta}$ are determined. We note that the vector potential as used here is in the full model consisting of both active and inactive coordinates. In the *normal-form coordinate basis*, the potential above for each player j, is the covariant vector δ^j_μ. In mathematical form, this is the statement above that the payoff is associated with the single 1-form $\mathbf{U^j}$ with unit coefficient.

As we develop the theory, a physical, geometric and global meaning of these transformations will be helpful. We make progress in this regard by looking at the implications of the transformations on the *acceleration potentials*, Eq. (1.63). Consider the set of directions along α in the co-moving frame and the corresponding covariant vectors $E_\mu{}^\alpha$. The transformation rule for the acceleration potentials leads to the following form for the covariant derivative:

$$E_{\mu;\nu}{}^\alpha = \partial_\nu E_\mu{}^\alpha - \varpi^\lambda{}_{\mu\nu} E_\lambda{}^\alpha = -\omega^\alpha{}_{\beta\gamma} E_\mu{}^\beta E_\nu{}^\gamma \tag{3.31}$$

We can use Eq. (3.31) to determine the orientation potential in one frame from the covariant derivative.

An alternate form is the inverse:

$$E^\mu{}_{\alpha\backslash\beta} = \Delta_\beta E^\mu{}_\alpha - \omega^\gamma{}_{\alpha\beta} E^\mu{}_\gamma = -\varpi^\mu{}_{\nu\lambda} E^\nu{}_\alpha E^\lambda{}_\beta \tag{3.32}$$

Here we consider the (inverse) transformation matrix $E^\mu{}_\alpha$ to be a vector field (***time-like*** and ***space-like***) in the *co-moving orthonormal coordinate basis* labeled by the normal-form dimensions μ. As noted, these expressions reflect the transformation properties of the acceleration potentials:

$$\omega^\alpha{}_{\beta\gamma} = \varpi^\lambda{}_{\mu\nu} E_\lambda{}^\alpha E^\mu{}_\beta E^\nu{}_\gamma - E^\mu{}_\beta E^\nu{}_\gamma \partial_\nu E_\mu{}^\alpha \tag{3.33}$$

Equivalently, we have the inverse relationship:

$$\varpi^{\mu}{}_{\nu\lambda} = \omega^{\gamma}{}_{\alpha\beta} E^{\mu}{}_{\gamma} E_{\nu}{}^{\alpha} E_{\lambda}{}^{\beta} - E_{\nu}{}^{\alpha} E_{\lambda}{}^{\beta} \Delta_{\beta} E^{\mu}{}_{\alpha} \tag{3.34}$$

These equations provide helpful relations as we formulate models.

In the *normal-form coordinate basis*, we expressed the field equations based on the ***harmonic gauge***. We finish this section by expressing the *harmonic gauge* condition and exactness statements in the *co-moving basis*. The consequence of the *harmonic gauge* follows from Eq. (3.31), writing the active and inactive metrics out explicitly using the normal-form potentials:

$$\begin{aligned} g^{ab} E^{\alpha}{}_{a;b} + \gamma^{jk} E^{\alpha}{}_{j;k} &= -\omega^{\alpha\beta}{}_{\beta} \\ \frac{1}{\sqrt{\gamma g}} \partial_a \left(\sqrt{\gamma g}\, g^{ab} E^{\alpha}{}_{b} \right) &= -\omega^{\alpha\beta}{}_{\beta} \end{aligned} \tag{3.35}$$

We use the *harmonic gauge* condition Eq. (3.13) to obtain:

$$g^{ab} \partial_a E^{\alpha}{}_{b} = -\omega^{\alpha\beta}{}_{\beta} \tag{3.36}$$

We will be interested in specifying frames that are torsion free, Eq. (1.54), that are orthonormal so that g_{ab} is determined by the frames from Eq. (3.27), that have the orientation potentials determined by the field equations, that satisfy the frame equations Eq. (1.54) and that satisfy the ***wave equation*** Eq. (3.36). The advantage of this *wave equation* is that it is formulated in terms of the *exact* active strategies x^a in the *normal-form coordinate basis*.

As an example of the use of this *wave equation*, consider a model in which there is a direction in the co-moving orthonormal coordinate basis $\mathbf{E}^{\upsilon}$ that is also exact. Applying Eq. (3.3) to this, we have the condition $E_{\mu;\lambda}{}^{\upsilon} = E_{\lambda;\mu}{}^{\upsilon}$, which has the following solution:

$$\begin{aligned} \partial_a E_j{}^{\upsilon} &= 0 \\ \partial_a E_b{}^{\upsilon} &= \partial_b E_a{}^{\upsilon} \end{aligned} \tag{3.37}$$

The first equation requires each component to be constant; we pick the gauge in which the constant is zero, $E_j{}^{\upsilon} = 0$. Based on the second equation, there will be a potential field y^{υ} whose constant surfaces define a coordinate and whose associated vector field is determined by the potential, $E_a{}^{\upsilon} = \partial_a y^{\upsilon}$. That the space might be curved will limit the number of such coordinates. However when there is such a coordinate, the condition Eq. (3.35) provides the wave equation:

$$g^{ab} \partial_a \partial_b y^{\upsilon} = -\omega^{\upsilon\beta}{}_{\beta} \tag{3.38}$$

If the metric is known in terms of the ***harmonic coordinates*** and if the same is true of the orientation potentials, this equation provides the coordinate potential y^{υ} in terms of the *harmonic coordinates*.

An alternate and useful approach is to couple Eq. (3.32) with the *harmonic gauge condition*, considering the active and inactive dimensions separately:

$$\begin{aligned} m^{\alpha\beta} E^{a}{}_{\alpha|\beta} &= -\varpi^{a}{}_{bc} g^{bc} - \varpi^{a}{}_{kl} \gamma^{kl} = \frac{1}{\sqrt{\gamma g}} \partial_b \left(\sqrt{\gamma g}\, g^{ab} \right) = 0 \\ m^{\alpha\beta} E^{j}{}_{\alpha|\beta} &= -\varpi^{j}{}_{bc} g^{bc} - \varpi^{j}{}_{kl} \gamma^{kl} = 0 \end{aligned} \tag{3.39}$$

The active components are zero because of the *harmonic gauge condition* Eq. (3.13) and the inactive components are zero because of the properties of Eq. (1.74). Because the active directions are ***exact***, we have in addition, Eq. (3.3):

$$E^{a}{}_{\alpha|\beta} = E^{a}{}_{\beta|\alpha} \tag{3.40}$$

This determines the exactness condition in the co-moving frame in terms of the orientation potentials. In this frame there is no guarantee that the differential operators commute, so we don't expect that in this frame the transformation matrix will be expressed in terms of the gradients of a potential field. This is an example where our intuition from Newtonian physics is not helpful. We anticipate that the directions that don't commute will be the flow direction with any direction whenever there is *frame rotation*.

The geometric significance of the influence of *frame rotation* can be illustrated with the simple example of a sphere (Gockeler & Schucker, 1987), where the latitude and longitude are holonomic coordinates $\{\theta,\phi\}$ but they are not orthonormal. The coordinates on a sphere that are orthonormal are $\{\sin\theta d\phi, d\theta\}$, however these coordinates are not holonomic. From Einstein's theory, time and a space direction $\{t,u\}$ may be holonomic but not orthonormal. The coordinates that might be orthonormal are $\{\sqrt{g_{tt}}dt, du\}$, but in general these will not be holonomic because of *frame rotation* effects.

We will have occasion to use both types of frames in subsequent chapters. For example we distinguish the ***co-moving basis*** as short for the ***co-moving orthonormal coordinate basis.*** This frame in general is not holonomic. Following the nomenclature and previous analysis (Thomas G. H., 2006), there is also a ***central holonomic frame*** (*Cf.* section 4.8, exercise 53). In general this frame will not be orthonormal. We extend the previous analysis by solving the field equations for a class of models whose behaviors will be displayed in both frames.

3.7 *Decomposition of the orientation potentials* $\omega^{\gamma}{}_{\alpha\beta}$

We have a general *decision process theory* that leads to field equations Eq. (3.1) that allow us to investigate decision processes using the scientific method to expand our understanding and our approach. We have players identified in a persistent way with inactive strategies along with the decisions they make that are active. We treat time as an active dimension. We have argued that there always will be a *time-like* unit vector that describes the flow of energy as well as a metric tensor such that these vectors and tensors depend only on the active strategies available to each agent. Decisions occur in this space-time x^a that ranges over time and the active strategies. In this *normal-form coordinate basis*, none of the tensors depend on the inactive space components ξ^j. In this frame, the metric components between the active and inactive space are zero. In the last section, we extended this general discussion to the *co-moving orthonormal coordinate basis*.

The vector Eq. (2.37) representing the flow of energy occupies a central role in the theory. Some of the orientation potentials related to the flow in the co-moving frame have large scale geometric significance that makes it easier to understand the anticipated behaviors. We take one of the orthonormal coordinate basis vectors to be the flow of energy $E_\mu{}^o = V_\mu$. We indicate this dimension by the Greek letter o.

The following choices have proved insightful in the study of fluids. We believe they will be helpful in the study of decision processes. The application of the transformation rule Eq. (3.31) can be used to define an antisymmetric ***vorticity tensor*** $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, a symmetric ***expansion tensor*** $\theta_{\alpha\beta} = \theta_{\beta\alpha}$ and an ***acceleration vector*** q^α:

$$\begin{aligned}
E_{\mu;\nu}{}^o &= V_{\mu;\nu} = \omega_{\alpha\beta} E_\mu{}^\alpha E_\nu{}^\beta + \theta_{\alpha\beta} E_\mu{}^\alpha E_\nu{}^\beta + q_\alpha E_\mu{}^\alpha V_\nu \\
\omega^o{}_{\alpha\beta} &= -\omega_{\alpha\beta} - \theta_{\alpha\beta} \\
\omega^o{}_{\alpha o} &= -q_\alpha \\
\omega^o{}_{o\alpha} &= 0
\end{aligned} \tag{3.41}$$

There is no term proportional to $V_\mu E_\nu{}^\alpha$ because unit flow is orthogonal to its covariant gradient; equivalently we use the antisymmetric nature of the orientation potential Eq. (1.60) in the first two indices to prove this.

Thinking of the flow as a fluid, one considers a unit cube of fluid and follows its path in the co-moving frame. The cube can rotate where the rotation is specified by the ***vorticity tensor*** and the cube can expand, contract or shear as specified by the ***expansion tensor***. The acceleration of the cube is given by q^{α} in the co-moving coordinate basis since

$$\frac{DV_{\mu}}{\partial\tau} = V_{\mu;\nu}V^{\nu} = q_{\alpha}E_{\mu}{}^{\alpha} \tag{3.42}$$

Because the orientation potentials are antisymmetric in the first two indices, these same parameters determine $\omega_{\alpha o \beta}$, so that an expansion of a vector transverse to the flow is:

$$E_{\mu;\nu}{}^{\alpha} = -q^{\alpha}V_{\mu}V_{\nu} - \left(\omega^{\alpha}{}_{\beta} + \theta^{\alpha}{}_{\beta}\right)V_{\mu}E_{\nu}{}^{\beta} + \phi^{\alpha}{}_{\beta}E_{\mu}{}^{\beta}V_{\nu} - \omega^{\alpha}{}_{\beta\gamma}E_{\mu}{}^{\beta}E_{\nu}{}^{\gamma} \tag{3.43}$$

In this expression none of the indices are along the flow, $\alpha,\beta,\gamma,\dots \neq o$. We define a rate of change of the transverse vector as

$$\frac{D_F E_{\mu}{}^{\alpha}}{\partial\tau} = \frac{DE_{\mu}{}^{\alpha}}{\partial\tau} + q^{\alpha}V_{\mu} = \phi^{\alpha}{}_{\beta}E_{\mu}{}^{\beta} \tag{3.44}$$

We have introduced the Fermi derivative (Hawking & Ellis, 1973) to define the rate of change:

$$\frac{D_F X^{\mu}}{\partial\tau} \equiv \frac{DX^{\mu}}{\partial\tau} + \frac{DV^{\nu}}{\partial\tau}X_{\nu}V^{\mu} - V^{\nu}X_{\nu}\frac{DV^{\mu}}{\partial\tau} \tag{3.45}$$

This rate of change, which is defined in terms of the covariant derivative, has the following useful properties that help us understand the geometric meaning of the orientation potentials. Any pair of vector fields that have zero Fermi derivative will maintain lengths and angle. The Fermi derivative measures the amount of rotation of the basis. Using the Fermi derivative, we deduce that the significance of the antisymmetric orientation potential coefficients $\phi_{\alpha\beta} = -\omega_{\alpha\beta o} = -\phi_{\beta\alpha}$ is that they specify the ***frame rotation*** along the path.

We note that transformation to the flow vector has both active and inactive components V^{a} and V^{j}, which is a transformation to a vector field whose 1-form does not in general vanish. In other words, the flow corresponds in general to a 1-form that is not *exact*. In Einstein's theory of relativity, this is stated as the twin paradox: twins travelling along different paths may age differently as a result of their paths seeing different accelerations: the paths are not ***inertial***. This is true in this theory as well. This makes sense because we expect the flow to reflect the influence of sources that generate acceleration. We address the question about whether the transverse components of the orthonormal set need be *exact* in the next section.

3.8 *Player fixed frame model*

We see that a key characteristic of behaviors in this theory will be the ***vorticity*** and frame rotations. We focus our study using a specific model in which the *expansion tensor* plays little role. We recall our assumption that space-time locally has the property that in an open set of any local region, the distinction that strategies have values is expressed mathematically by the existence of a scalar function x^{a} that defines a hyper surface in that region (of dimension one less than the total dimension of the space-time). We say that the coordinate 1-form is ***exact***, meaning its 1-form vanishes, $d\mathbf{U}^{\boldsymbol{\alpha}} = 0$. In the *normal-form coordinate basis*, the inactive strategies are not *exact* whereas time and the active strategies are *exact*.

Let us suppose that there are n_a active strategies in the *normal-form coordinate basis*, n_i inactive strategies or players giving a total of $n = n_a + n_i$ ***space dimensions***. We consider a wide class of models formulated in the *co-moving orthonormal coordinate basis*, which we call the ***player fixed frame model,*** in which there are n_a ***proper-active*** strategies y^{υ} in which the corresponding 1-forms are exact. We denote these active strategies by the Greek letters $\{\upsilon,\upsilon',\upsilon'',\dots\}$. In addition to the ***proper time coordinate*** $\{o\}$, which is the coordinate along the flow and these proper-active strategies $\{\upsilon\}$, we assume the remaining n_i strategies are inactive and denote them by $\{\alpha,\beta,\gamma,\cdots\}$ and term them ***proper-inactive*** strategies. In general, the *proper time coordinate* and the *proper-inactive strategies* will not be *exact*.

This is a very broad class of models in that we allow any number of players, any number of strategies and any specification of inertial and applied forces. Though this class of models appears to us currently as natural, we will need to consider a broader class of models as we gain better tools with which to solve the equations. For now however, this class of models we think of sufficient interest to form the basis for quantitative examples in this book.

We analyze the consequences that the proper-active strategies are *exact* Eq. (3.37). We also write in the *co-moving basis*, the condition that the inactive strategies satisfy the Killing conditions Eq. (3.22). The first set of consequences follows from the assumption of *exactness*, Eq. (3.37):

$$\begin{aligned} &E_j^{\ \upsilon} = 0 \\ &\partial_a E_b^{\ \upsilon} = \partial_b E_a^{\ \upsilon} \end{aligned} \tag{3.46}$$

The consequence of the *exactness* condition $d\mathbf{E}^{\upsilon} = 0$ can be imposed also on the torsion free requirement Eq. (1.54):

$$\begin{aligned} &\alpha,\beta,\gamma \neq o,\upsilon \Rightarrow \\ &d\mathbf{E}^{\upsilon} = \begin{pmatrix} \left(\omega^{\upsilon}{}_{\upsilon' o} - \omega^{\upsilon}{}_{o\upsilon'}\right)\mathbf{E}^{\upsilon'}\wedge\mathbf{E}^{o} + \omega^{\upsilon}{}_{\upsilon'\upsilon''}\mathbf{E}^{\upsilon'}\wedge\mathbf{E}^{\upsilon''} \\ +\left(\omega^{\upsilon}{}_{o\alpha} - \omega^{\upsilon}{}_{\alpha o}\right)\mathbf{E}^{o}\wedge\mathbf{E}^{\alpha} + \left(\omega^{\upsilon}{}_{\upsilon'\alpha} - \omega^{\upsilon}{}_{\alpha\upsilon'}\right)\mathbf{E}^{\upsilon'}\wedge\mathbf{E}^{\alpha} + \omega^{\upsilon}{}_{\alpha\beta}\mathbf{E}^{\alpha}\wedge\mathbf{E}^{\beta} \end{pmatrix} = 0 \end{aligned} \tag{3.47}$$

We have expanded the wedge product using the notation from the previous section, where we separate the behaviors along the energy flow direction, the proper-active strategy directions and the remaining proper-inactive strategy directions.

We convert the consequence of *exactness* to conditions on the orientation potentials. We write these orientation potentials using the decompositions Eq. (3.41) and (3.43):

$$\begin{aligned} &-\phi^{\upsilon}{}_{\upsilon'} - \omega^{\upsilon}{}_{\upsilon'} - \theta^{\upsilon}{}_{\upsilon'} = 0 \\ &\omega^{\upsilon}{}_{\alpha} + \theta^{\upsilon}{}_{\alpha} + \phi^{\upsilon}{}_{\alpha} = 0 \\ &\omega^{\upsilon}{}_{\upsilon'\upsilon''} = \omega^{\upsilon}{}_{\upsilon''\upsilon'} \\ &\omega^{\upsilon}{}_{\upsilon'\alpha} - \omega^{\upsilon}{}_{\alpha\upsilon'} = 0 \\ &\omega^{\upsilon}{}_{\alpha\beta} = \omega^{\upsilon}{}_{\beta\alpha} \end{aligned} \tag{3.48}$$

With this notation the five distinct terms give five sets of conditions, which can be simplified:

$$\begin{aligned} &\phi_{\upsilon\upsilon'} = -\omega_{\upsilon\upsilon'} \\ &\theta_{\upsilon\upsilon'} = 0 \\ &\omega_{\upsilon\alpha} + \theta_{\upsilon\alpha} + \phi_{\upsilon\alpha} = 0 \\ &\omega_{\upsilon\upsilon'\upsilon''} = 0 \\ &\omega_{\upsilon\upsilon'\alpha} = \omega_{\upsilon\alpha\upsilon'} = -\omega_{\alpha\upsilon\upsilon'} = \omega_{\alpha\upsilon'\upsilon} = -\omega_{\upsilon'\upsilon\alpha} \\ &\omega^{\upsilon}{}_{\alpha\beta} = \omega^{\upsilon}{}_{\beta\alpha} \end{aligned} \tag{3.49}$$

These conditions along with the additional conditions Eq. (3.46) on the frame implies that there is a coordinate potential y^{υ} that is a scalar function of the *harmonic coordinates* and $E_a^{\ \upsilon} = \partial_a y^{\upsilon}$.

We add the following additional assumptions to the *player fixed frame model* that we have found by trial and error to be consistent and provide us a subclass of solvable models that articulate one notion of large scale persistency:

$$q^{\alpha} = 0$$
$$\omega_{\alpha\beta\upsilon} = \theta_{\alpha\beta} = \omega_{\alpha\beta\gamma} = 0 \qquad (3.50)$$
$$\phi_{\alpha\beta} = -\omega_{\alpha\beta} = 0$$

Unless otherwise stated, we take this subclass to be the *player fixed frame model*, though we can't exclude other possibilities. We will justify that these conditions are indeed consistent and articulate this notion of persistency. To obtain further relationships, we pursue the consequences of persistency as viewed in the co-moving basis. We do that in the next section.

3.9 *Persistency under the player fixed frame model*

In the last section we took a step closer to what we claim will be a complete set of numerically solvable equations with which to determine the properties of the ***decision process theory*** based on a general theoretical framework. We articulated an approach by considering a wide class of models, the ***player fixed frame model***.

That we expect these models are solvable numerically requires some explanation. There are theorems that show that the field equations Eq. (3.1) have solutions for any model. The general theorems don't yet provide tractable ways of doing the numerical analysis for these coupled non-linear partial differential equations for all models. Instead, solvable models based on simplifications are constructed that help one to understand particular features of the theory. What we would like to do is find such a set of solvable models. We believe the current approach comes close to meeting these criteria, as well as providing additional insight into the meaning of the equations. To proceed we have three remaining steps.

- We impose the consistency of the orthogonality relations Eq. (3.27).
- We analyze the Killing relations Eq. (3.22).
- We provide the field equations Eq. (3.1) in the co-moving orthonormal coordinate basis.

In this section we address the Killing relations. In the next chapter we address the field equations and provide a set of solvable models in section 4.5.

We flesh out the consequences based on the *player fixed frame model* from the orthogonality relations (3.27) for the mixed tensor components that are zero:

$$E_{ao}E_j^{\ o} + E_{a\alpha}E_j^{\ \alpha} = 0 \qquad (3.51)$$

There are no terms with $E_j^{\ \upsilon}$ since these transformation components are zero because the exactness condition of $\mathbf{E}^{\upsilon}$ implies Eq. (3.46). Implicit is our assumption that the matrix $E_j^{\ \alpha}$ is not singular, which implies that the transformation vector $E_{a\alpha}$ is proportional to E_{ao}:

$$E_{a\alpha} = e_{\alpha}E_{ao} \qquad (3.52)$$

Substituting this into the expression Eq. (3.51), we get a projection of the flow onto the inactive directions:

$$E_{ao}\left(E_j^{\ o} + e_{\alpha}E_j^{\ \alpha}\right) = 0 \qquad (3.53)$$

Because at least one component of the timelike flow is not zero, the implication is:

$$E_j^{\ o} = -e_{\alpha}E_j^{\ \alpha} \qquad (3.54)$$

A consequence of this rule is that if we interpret $E_j^{\ o}$ as the charge for player j, then e_{α} is the ***proper charge*** for *proper-player* α.

The orthogonality relations also demonstrate that the derivative operator associated with the proper inactive strategy is not zero but proportional to the proper time frame derivative:

$$\Delta_{\alpha} = E^{a}_{\ \alpha}\partial_a = e_{\alpha}\Delta_o \qquad (3.55)$$

We call Δ_o the variation along the flow, the variation along the ***proper time coordinate***. We also see that the operator ∂_j is zero based on Eq. (3.54), as it should be on the space of functions of interest:

$$\partial_j = \left(E_j^{\ o} E^a{}_o + E_j^{\ \alpha} e_\alpha E^a{}_o \right) \partial_a = 0 \tag{3.56}$$

We get further information on the charge by considering the consequences of gauge invariance using the *wave equation* Eq. (3.38) and its companion Eq. (3.36) for the flow:

$$\begin{aligned} g^{ab}\partial_a \partial_b y^\upsilon &= -q^\upsilon - \omega^{\upsilon\beta}{}_\beta \\ g^{ab}\partial_a E^o{}_b &= 0 \end{aligned} \tag{3.57}$$

Here we make use of the assumptions above concerning the values of the orientation potentials. We apply Eq. (3.52) to the gauge condition Eq. (3.36), and get:

$$g^{ab}\partial_b E_a^{\ \alpha} = g^{ab}\partial_b \left(e_\alpha E_a^{\ o} \right) = 0 \tag{3.58}$$

This shows that the "charge" is conserved along a streamline.

$$\begin{aligned} &0 = g^{ab}\partial_b \left(e_\alpha E_a^{\ o} \right) = e_\alpha g^{ab}\partial_b E_a^{\ o} + g^{ab} E_a^{\ o} \partial_b e_\alpha = e_\alpha g^{ab}\partial_b E_a^{\ o} + \Delta_o e^\alpha \\ &\Rightarrow \Delta_o e^\alpha = 0 \end{aligned} \tag{3.59}$$

The result follows directly from the second line in Eq. (3.57).

Since the coordinates x^a are *exact*, this coordinate vector $E^a{}_\alpha$ in a new coordinate system satisfies Eq. (3.40): $E^a{}_{\alpha|\beta} = E^a{}_{\beta|\alpha}$. For the *player fixed frame model*, we apply these conditions and find that there are two distinct classes of equations:

$$\begin{aligned} \Delta_o E^a{}_\upsilon - \Delta_\upsilon E^a{}_o &= -q_\upsilon E^a{}_o - 2\theta_\upsilon^{\ \alpha} e_\alpha E^a{}_o \\ \Delta_\upsilon E^a{}_{\upsilon'} - \Delta_{\upsilon'} E^a{}_\upsilon &= \left(2\omega_{\upsilon\upsilon'} - 2\omega^\alpha{}_{\upsilon\upsilon'} e_\alpha \right) E^a{}_o \end{aligned} \tag{3.60}$$

This shows that in this basis, the vector fields are determined. The first equation determines the "curl" of the vector field in terms of the acceleration and what we show below is the electric field in the co-moving frame. The second equation determines the "curl" in terms of what we show later to be the payoff matrix in the co-moving frame.

We have determined equations for all the vector fields $\mathbf{E^a}$ associated with the active components. We now turn to the vector fields $\mathbf{E_j}$ associated with the inactive components. In the *normal-form coordinate basis*, the Killing vectors are the inactive vectors that lie along each of the coordinate directions. So for example the Killing vector field $\bar{K}(j)^\mu$ along the j direction has components $\bar{K}(j)^\mu = \delta_j^\mu$. We transform this to the co-moving basis to get:

$$K(j)^\rho = E_\mu^{\ \rho} \bar{K}(j)^\mu = E_j^{\ \rho} \tag{3.61}$$

We interpret the transformation field $E_j^{\ \rho}$ for each fixed inactive strategy j to be the Killing vector field in the co-moving frame.

Given our decomposition of coordinates into proper time, proper-active and proper-inactive, there are six cases of the Killing relation Eq. (3.22) to consider, using the notation of the previous section that $\alpha, \beta, \gamma \neq o, \upsilon$:

$$\begin{aligned} E_{j\alpha|\beta} + E_{j\beta|\alpha} &= 0 \\ E_{j\upsilon|o} + E_{jo|\upsilon} &= 0 \\ E_{j\alpha|o} + E_{jo|\alpha} &= 0 \\ E_{j\upsilon|\upsilon'} + E_{j\upsilon'|\upsilon} &= 0 \\ E_{j\alpha|\upsilon} + E_{j\upsilon|\alpha} &= 0 \\ E_{jo|o} + E_{jo|o} &= 0 \end{aligned} \tag{3.62}$$

We summarize the result for each case in order, leaving to the exercises at the end of the chapter the derivations.

1. We start with the first case and write out the covariant derivative in terms of the orientation potentials with the *player fixed frame model* that the transformations depend only on the *proper time coordinate* and proper-active strategies:
$$e_\beta \Delta_o E_{j\alpha} + e_\alpha \Delta_o E_{j\beta} = 0 \tag{3.63}$$
2. The second case provides the proper time variation of the proper-active strategy of the Killing vector field components:
$$\Delta_\upsilon E_{jo} = -q_\upsilon E_{jo} - 2\omega_\upsilon{}^\alpha E_{j\alpha} \tag{3.64}$$
3. The third case provides the time variation of the proper-inactive strategies of the Killing vector field components to be zero and shows that Eq. (3.63) is identically satisfied:
$$\Delta_o E_{j\alpha} = 0 \tag{3.65}$$
4. The fourth case has no time derivatives and is identically satisfied based on Eq. (3.46):
$$\Delta_\upsilon E_{j\upsilon'} + \Delta_{\upsilon'} E_{j\upsilon} = 0 \tag{3.66}$$
5. The fifth equation gives the spatial dependence of the proper-inactive components:
$$\Delta_\upsilon E_{j\alpha} = -2\theta_{\alpha\upsilon} E_{jo} - \omega_{\upsilon\alpha}{}^\beta E_{j\beta} \tag{3.67}$$
6. The sixth case is the time derivative of the component of the Killing vectors along the energy flow:
$$\Delta_o E_{jo} = q^\alpha E_{j\alpha} + q^\upsilon E_{j\upsilon} = 0 \tag{3.68}$$
7. The last equation due to Eq. (3.54) determines the "charge" gradient:
$$\Delta_\upsilon e_\alpha = -q_\upsilon e_\alpha + 2\omega_{\upsilon\alpha} - 2\theta_{\beta\upsilon} e_\alpha e^\beta + \omega_{\upsilon\alpha\beta} e^\beta \tag{3.69}$$

The important overall result is that we get first order partial differential ***gradient*** equations for the proper-time evolution of the flow component, proper-active components and proper-inactive components of each Killing vector field. We get two sets of constraint equations involving gradients and one algebraic equation. The frame transformations $\{E_{j\alpha}, \quad E_{jo}\}$ are independent of proper time and the remaining components $E_{j\upsilon}$ are zero.

These equations determine the inactive metric and show that it is independent of proper time, since each of the components is similarly independent:
$$\gamma_{jk} = E_{jo} m^{oo} E_{ko} + E_{j\alpha} m^{\alpha\beta} E_{k\beta} \tag{3.70}$$
From the *harmonic gauge conditions* Eq. (3.39) and exactness conditions Eq. (3.60) we obtain the equations that determine $E^a{}_o \quad E^a{}_\upsilon$:
$$\begin{aligned} \left(1 + e_\alpha e^\alpha\right) \Delta_o E^{ao} &= -\Delta_\upsilon E^{a\upsilon} + \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) E^a{}_\upsilon \\ \Delta_o E^a{}_\upsilon &= \Delta_\upsilon E^a{}_o - q_\upsilon E^a{}_o - 2\theta_\upsilon{}^\alpha e_\alpha E^a{}_o \end{aligned} \tag{3.71}$$
Given this solution to Eq. (3.71), the active metric is determined:
$$g^{ab} = \left(1 + e_\alpha e^\alpha\right) E^a{}_o E^{bo} + E^a{}_\upsilon E^{b\upsilon} \tag{3.72}$$
The active metric may depend on proper-time. Even in the case that it is independent of proper-time, it does depend on the acceleration and so there will be effects due to acceleration in the solution.

We have pointed out previously, Eq. (3.30), that the co-moving payoff is determined once the transformations are known. We can also proceed as follows to determine the payoff. The payoff matrix is an acceleration-potential in the *normal-form coordinate basis*, Eq. (1.74) as seen for example in ϖ_{jab}. Since the orientation potentials are related, we us Eq. (3.32) to determine the payoff fields:
$$\tfrac{1}{2} F^j{}_{ab} = \begin{pmatrix} \omega_{\alpha\upsilon\upsilon'} E_a{}^\upsilon E_b{}^{\upsilon'} E^{j\alpha} - \omega_{\upsilon\upsilon'} E_a{}^\upsilon E_b{}^{\upsilon'} E^{jo} \\ + \left(q_\upsilon E^{jo} + \left(\omega_{\upsilon\alpha} + \theta_{\upsilon\alpha}\right)\left(E^{j\alpha} + e^\alpha E^{jo}\right) + \omega_{\upsilon\alpha\beta} e^\alpha E^{j\beta}\right)\left(E_a{}^o E_b{}^\upsilon - E_a{}^\upsilon E_b{}^o\right) \end{pmatrix} \tag{3.73}$$

The expression is informative. It shows that terms such as $\omega_{\alpha\upsilon\upsilon'}$ determine the payoff matrix and that the electric field is determined by $\omega_{\upsilon\alpha}+\theta_{\upsilon\alpha}$. This shows that we are on the right track and that we can gain insight into the equations in the co-moving frame. From this form we determine also the various co-moving payoff components:

$$\begin{aligned}
&f^{j}{}_{o\alpha}=0\\
&f^{j}{}_{\alpha\beta}=0\\
&f^{j}{}_{\alpha\upsilon}=e_{\alpha}f^{j}{}_{o\upsilon}\\
&f^{j}{}_{o\upsilon}=2\left(q_{\upsilon}E^{jo}+\left(\omega_{\upsilon\alpha}+\theta_{\upsilon\alpha}\right)\left(E^{j\alpha}+e^{\alpha}E^{jo}\right)+\omega_{\upsilon\alpha\beta}e^{\alpha}E^{j\beta}\right)\left(1-E^{k}{}_{o}E_{k}{}^{o}\right)\\
&f^{j}{}_{\upsilon\upsilon'}=2\omega_{\alpha\upsilon\upsilon'}E^{j\alpha}-2\omega_{\upsilon\upsilon'}E^{jo}
\end{aligned}\tag{3.74}$$

Zero payoffs in the co-moving basis are required between *proper-inactive strategies* and the *proper time coordinate* and between *proper-inactive strategies*.

We need the field equations using Eq. (3.1) and its explicit form in terms of orientation potentials Eq. (1.67) applied to the co-moving frame to provide the complete set of equations. The field equations will provide first order partial differential equations for the orientation potentials. We examine these equations in the next chapter, where we see the vorticity effects are made explicit as well as additional gradient effects. We show there to what extent we can reduce the equations to linear coupled differential equations that provide the underlying mechanisms for ***vorticity behaviors*** and ***gradient behaviors*** of decision processes.

3.10 *Outcomes*

After completing this chapter, the student should understand that persistency in *decision process theory* defines what it means to be an agent or player. The underlying mathematical structure is the group of isometries, showing the relationship between agents, players and measurement. There are quantitative consequences from this group of isometries that are expressed by differential equations, which have unique solutions in the *harmonic gauge*, given initial conditions specified on a space like hypersurface orthogonal to time. The equations have analogs to equations from physics, though these equations describe a different space, time and phenomena. The student should learn that the large scale behavior of the theory differs from the local behaviors because of the possibility of acceleration: *i.e.* dynamic changes of strategic choices. We have approached the large scale behaviors by introducing the *player fixed frame model* in both the *normal-form coordinate basis* and the *co-moving coordinate basis*. This model refines our notion of large scale persistency in a way that is consistent with and follows from the local principles of the theory.

The attainment of the outcomes is facilitated by doing the exercises at the end of this chapter. To further help the reader, we list more detailed outcomes from this chapter by section.

- From section 3.1, the student will have learned that decision process events at each point in time can be described locally in a coordinate (holonomic) basis in which each coordinate is a surface of constant potential for a potential field whose normal provides the coordinate direction. This underlies the assumption that we can specify the distance between any two decision process events at separate points in time.
- In section 3.2, the student learns that in the *normal-form coordinate basis,* gauge invariant results are directly obtained. This provides an example of a "global" basis that has advantages over the "local" coordinate basis. Though not holonomic, the basis is effectively holonomic when transformations are restricted to the active subspace.
- In section 3.3, we frame the field equations in the *normal-form coordinate basis* using the *harmonic gauge* that removes the gauge degrees of freedom. These field equations form the basis for the quantitative predictions of *decision process theory*.

- In section 3.4, the field equations that result are derived, supported by results from the exercises. These equations provide the student a guide to the behaviors expected in the theory and can observe the similarities and differences between *decision process theory* and physical theories.
- In section 3.5 the student will have learned that the Killing relations from differential geometry provide a covariant articulation of *isometry*, which is a common group structure at each point of space and time. In *decision process theory*, the set of players form the basis of this symmetry and so are persistent attributes of any solution.
- In section 3.6, the student will be introduced to the *co-moving orthonormal coordinate basis.* The student should be able to transform between this basis and the *normal-form coordinate basis* and understand how to express the covariant derivatives, exactness conditions and *harmonic gauge* conditions in each basis.
- In section 3.7 as preparation for creating models in the *co-moving basis*, a geometric decomposition of the orientation potentials is given. Certain of the orientation potentials describe how the frame rotates when moving along particular directions.
- In section 3.8 the *player fixed frame model* is defined and the consequence of exactness developed.
- In section 3.9, the consequences of persistency are developed, along with the further constraints of the commutation constraints and *harmonic gauge* constraints. The student should understand the basis of this class of models and the defining equations of this model in the theory.

3.11 *Exercises*

1. Using the transformation to *normal-form coordinates* Eq. (1.71), show that the Lie product of any two inactive vectors is zero, $\left[\mathbf{U_j},\mathbf{U_k}\right]=0$.
2. Using the transformation to *normal-form coordinates* Eq. (1.71), show that the Lie product of an active and an inactive vector is zero, $\left[\mathbf{U_a},\mathbf{U_k}\right]=0$.
3. Using the transformation to *normal-form coordinates* Eq. (1.71), show that the non-zero components of the Lie product of any pair of active vectors are $\left[\mathbf{U_a},\mathbf{U_b}\right]^j=-F^j{}_{ab}$, which correspond only to gradients in the inactive space. If one restricts attention to fields that are independent of the inactive coordinates, then the active vectors commute.
4. Use the covariance property of the "covariant derivative" to develop the formulae Eq. (3.31) and (3.32). Further, show for fixed choices of coordinate values $\{\mu,\ \nu\}$ that the transformations $\{E_\mu{}^\alpha,\ \ E_\nu{}^\beta\}$ represent the corresponding coordinate directions in the new frame. Show that their Lie product is:
$$\left[\mathbf{E_\mu},\mathbf{E_\nu}\right]^\alpha=-\left(\varpi^\lambda{}_{\mu\nu}-\varpi^\lambda{}_{\nu\mu}\right)E_\lambda{}^\alpha \tag{3.75}$$
5. Using Eq. (3.75), show that the Lie product of the coordinate directions vanishes if and only if the orientation potentials $\varpi^\lambda{}_{\mu\nu}$ are symmetric in the last two indices. Using the inverse transformation below, what do you conclude about the Lie product of the transformed coordinate directions:
$$\left[\mathbf{E_\alpha},\mathbf{E_\beta}\right]^\mu=-\left(\omega^\gamma{}_{\alpha\beta}-\omega^\gamma{}_{\beta\alpha}\right)E_\gamma{}^\mu \tag{3.76}$$
6. Demonstrate the following form for the curvature tensor components from $\mathfrak{R}^a{}_b$ Eq. (2.28), where $\bar{R}_{abcd}$ are the curvature components as computed in a geometry of only the active coordinates:

$$\begin{aligned} R_{abcd} &= \bar{R}_{abcd} - \tfrac{1}{2}\gamma_{jk} F^j_{ab} F^k{}_{cd} - \tfrac{1}{4}\gamma_{jk} F^j_{ac} F^k_{bd} + \tfrac{1}{4}\gamma_{jk} F^j_{bc} F^k_{ad} \\ R_{abcj} &= -\tfrac{1}{4}\partial_b \gamma_{kj} F^k_{ac} + \tfrac{1}{4} F^k_{bc} \partial_a \gamma_{jk} - \tfrac{1}{2}\left(\gamma_{jk} F^k_{ab}\right)_{;c} \\ R_{abjk} &= \tfrac{1}{4}\left(\gamma_{jl}\gamma_{km} F^l_{ac} g^{cd} F^m_{db} - \gamma_{jl}\gamma_{km} F^l_{bc} g^{cd} F^m_{da} - \gamma^{ml}\partial_a \gamma_{jm} \partial_b \gamma_{kl} + \gamma^{ml}\partial_b \gamma_{jm}\partial_a \gamma_{kl}\right) \end{aligned} \quad (3.77)$$

7. Demonstrate the following form for the curvature tensor components from $\mathfrak{R}^i{}_j$ Eq. (2.28):

$$\begin{aligned} R^{ij}{}_{ab} &= \tfrac{1}{4}\gamma_{lk}\partial_a \gamma^{jk}\partial_b \gamma^{il} - \tfrac{1}{4}\gamma_{lk}\partial_a \gamma^{il}\partial_b \gamma^{jk} - \tfrac{1}{4} g^{cd} F^i_{ac} F^j_{bd} + \tfrac{1}{4} g^{cd} F^i_{bc} F^j_{ad} \\ R^{ijk}{}_b &= \tfrac{1}{4} g^{ac}\partial_a \gamma^{jk} F^j_{cb} - \tfrac{1}{4} g^{ac}\partial_c \gamma^{jk} F^i_{ab} \\ R_{ijlm} &= \tfrac{1}{4} g^{ab}\partial_a \gamma_{im}\partial_b \gamma_{jl} - \tfrac{1}{4} g^{ab}\partial_a \gamma_{il}\partial_b \gamma_{jm} \\ R^{ijkl} &= \tfrac{1}{4} g^{ab}\partial_a \gamma^{il}\partial_b \gamma^{jk} - \tfrac{1}{4} g^{ab}\partial_a \gamma^{ik}\partial_b \gamma^{lj} \end{aligned} \quad (3.78)$$

8. Demonstrate the following form for the curvature tensor components from $\mathfrak{R}^a{}_j$ Eq. (2.28):

$$\begin{aligned} R_{ajcd} &= -\tfrac{1}{4}\gamma_{jk;c} F^k_{ad} + \tfrac{1}{4}\gamma_{kj;d} F^k_{ac} - \tfrac{1}{2}\left(\gamma_{jk} F^k_{cd}\right)_{;a} \\ R_{ajbk} &= \tfrac{1}{4} g^{cd}\gamma_{jl}\gamma_{km} F^m_{ac} F^l_{bd} - \tfrac{1}{2}\gamma_{jk;ab} + \tfrac{1}{4}\gamma^{lm}\gamma_{kl;a}\gamma_{mj;b} \\ R_{ajlk} &= \tfrac{1}{4} g^{bd}\gamma_{lm}\gamma_{jk;d} F^m_{ab} - \tfrac{1}{4} g^{bd}\gamma_{km}\gamma_{jl;d} F^m_{ab} \end{aligned} \quad (3.79)$$

9. Using $R^j{}_{abc}$ from Eq. (3.79) and $R^j{}_{abc} + R^j{}_{bca} + R^j{}_{cab} = 0$ from Eq. (2.71), show Eq. (1.20), which implies that the payoff matrix is derivable from a potential.
10. Using the explicit forms from the previous exercises for the curvature tensor in the *normal-form coordinate basis*, show that the Einstein Eq. (3.1) yields the generalization of Maxwell Eq. (3.14) in the *harmonic gauge*. In a general gauge, show that the result is:

$$\tfrac{1}{2}\frac{1}{\sqrt{\gamma}}\left(\sqrt{\gamma}\gamma_{jk} g^{bc} F^k_{ab}\right)_{;c} = \kappa T_{ja} \quad (3.80)$$

11. Using the *harmonic gauge* and the additional "covariant" gauge condition Eq. (3.15), develop the wave equations for the player vector potentials Eq. (3.16) from Eq. (3.80).
12. Using the explicit forms from the previous exercises for the curvature tensor in the *normal-form coordinate basis*, show that the Einstein Eq. (3.1) yields the field equations Eq. (3.17) for the *active strategies*.
13. Using the explicit forms from the previous exercises for the curvature tensor in the *normal-form coordinate basis*, show that the Einstein Eq. (3.1) yields the field equations Eq. (3.21) for the *inactive strategies*.
14. Write and discuss the field equations in the absence of a matter field and with inactive and active metric components equal to their Minkowski (flat) values.
15. Show that Poisson's law, Eq. (3.18), results as a static, weak field limit of Eq. (3.17).
16. Demonstrate that the commutator (Lie product) of two Killing vectors is again a Killing vector
17. Show that the results Eq. (3.49) follow from Eq. (3.48).
18. Using the expression for the Lie product Eq. (3.76), for the *player fixed frame model* and using the notation for that model from section 3.8, show that the Lie product for co-moving basis *proper-active strategies* are:

$$\begin{aligned} \left[\mathbf{E}_\upsilon, \mathbf{E}_{\upsilon'}\right]^a &= 2\left(\omega_{\upsilon\upsilon'} - e^\alpha \omega_{\alpha\upsilon\upsilon'}\right) E^{ao} \\ \left[\mathbf{E}_\upsilon, \mathbf{E}_{\upsilon'}\right]^j &= 2\omega_{\upsilon\upsilon'} E^{jo} - 2\omega_{\alpha\upsilon\upsilon'} E^{j\alpha} \end{aligned} \quad (3.81)$$

19. Using the expression for the Lie product Eq. (3.76), for the *player fixed frame model* and using the notation for that model from section 3.8, show that the Lie products for co-moving basis *proper-active strategies* with the co-moving flow vector are:

$$\left[\mathbf{E_o},\mathbf{E_\upsilon}\right]^a = -\left(q_\upsilon + 2\theta_{\upsilon\alpha}e^\alpha\right)E^{ao}$$
$$\left[\mathbf{E_o},\mathbf{E_\upsilon}\right]^j = -q_\upsilon E^{jo} - 2\theta_{\upsilon\alpha}E^{j\alpha} \quad (3.82)$$

20. Using the expression for the Lie product Eq. (3.76), for the *player fixed frame model* and using the notation for that model from section 3.8, show that the Lie products for co-moving basis *proper-active strategies* with the *proper-inactive strategies* are:

$$\left[\mathbf{E_\upsilon},\mathbf{E_\alpha}\right]^a = -\omega_{\upsilon\alpha\beta}e^\beta E^{ao} - 2\omega_{\upsilon\alpha}E^{ao}$$
$$\left[\mathbf{E_\upsilon},\mathbf{E_\alpha}\right]^j = -\omega_{\upsilon\beta\alpha}E^{j\beta} - 2\omega_{\upsilon\alpha}E^{jo} \quad (3.83)$$

21. Using the expression for the Lie product Eq. (3.76), for the *player fixed frame model* and using the notation for that model from section 3.8, show that the Lie products for co-moving basis *proper-inactive strategies* and of these strategies with the co-moving flow are:

$$\left[\mathbf{E_\alpha},\mathbf{E_\beta}\right]^\mu = \left[\mathbf{E_o},\mathbf{E_\beta}\right]^\mu = 0 \quad (3.84)$$

22. Deduce the seven cases that are provided that follow from Eq. (3.62).
23. Derive the following for the gradients of the inactive metric:

$$\Delta_\upsilon\gamma_{jk} = -2q_\upsilon E_{jo}E_{ko} - 2\left(\omega_{\upsilon\alpha}+\theta_{\upsilon\alpha}\right)\left(E_j{}^\alpha E_{ko} + E_{jo}E_k{}^\alpha\right) - 2\omega_{\upsilon\alpha\beta}E_j{}^\alpha E_k{}^\beta$$
$$\Delta_o\gamma_{jk} = 0 \quad (3.85)$$