4 Stationary Harmonics

We have identified the mathematics of ***decision process theory***. We now need to identify the concepts and distinctions of that theory as applied to decisions. We turn to engineering for guidance. For example in electrical engineering, a large effort is devoted to learning about DC circuits since the concepts can then be applied to AC circuits. The general case of transient behavior can then be understood in that context. Similarly in mechanical engineering we can start with statics, and then proceed to a description of standing waves, steady-state waves and finally general dynamics. We would like to do the same for *decision process theory*.

We think stationary models may help illuminate how we use the *decision process theory* for engineering models in general. For those familiar with electromagnetic theory, we note that our approach is analogous to starting with magnetostatics in which the currents are static and the magnetic fields don't vary in time. We then extend such solutions to *harmonic* solutions (***phasors***). Such electrodynamics models provide significantly more insight than electrostatic models and with concepts of *phasors* (*harmonics*), provide an important step towards a full understanding of the theory. We go from DC circuits to AC circuits. The only step missing is the study of transient effects, which is also illuminated by this approach.

The simplest situation however is not strictly analogous to an electric circuit; however it does have a lot in common with transmission lines and travelling waves. In both cases it makes sense to work in a frame of reference in which the behavior is observed to be stationary. If we travel along with the wave, then there will be no observed flow. In this chapter we consider the case of decision processes that have that property: it is possible to find a co-moving frame of reference in which all of the dynamic attributes of the problem are stationary. Technically, this means it should be possible to identify a frame of reference that is co-moving, holonomic and one in which time and the inactive player strategies are persistent and mutually commuting. We could then use the principle of least action to provide us with the equations of motion. With sufficient numerical techniques we could then solve these equations.

There is a barrier however to understanding. There are a large number of equations, which represent a large number of concepts that we have not explored. The purpose of this chapter is to carry out this exploration. To achieve that goal, we find it useful to take a bottom up approach rather than a top down approach. We start with a specific class of models that leads to our goal, the player fixed frame models. We write the consequences of each of the equations of motion and see what distinctions are implied. This is a challenging task because it demands a fair amount of mathematical juggling. To help the reader follow the thread, much of the work is summarized in tables and the details are left as exercises. Toward the end of the chapter we return to the goal stated here and demonstrate that we have indeed solved the class of models of interest: models that have stationary wave behaviors.

The detailed plan of the chapter is as follows. We first rewrite the field equations in the ***co-moving orthonormal coordinate*** basis, or ***co-moving basis*** for short. We then review the frame equations in this co-moving basis, followed by the field equations making the assumptions of the ***player fixed frame model***. The basic results of the chapter are summarized in four tables, Table 4-1, Table 4-4, Table 4-5 and Table 4-6. The expressions for the frame are provided in section 4.1. In section 4.2 we show how we generated the field equations in the co-moving frame and in section 4.3, we simplify the expressions using the *player fixed frame model*. To interpret these results for *decision process theory*, in section 4.4 we articulate the distinctions that result from an examination of this model. Section 4.5 discusses these models in which the orientation potentials are *stationary*, though the coordinate wave equation produces active coordinates that are not. In an exercise, it is shown that models in this class can be written in a co-moving holonomic frame in which time is inactive. This justifies the assertion that there is a holonomic central frame in which time is inactive. With sufficient computing power, models in this class admit full numerical solutions using coupled partial differential equations, albeit with a rather large number (potentially hundreds) of variables. In section 4.6, we provide a sub-class of models that have a single active strategy. Models in this class admit full numerical solutions using coupled ordinary differential equations.

4.1 *Player Fixed Frame Model*

Before diving into the equations and the model, let's recall how we got to this point. We have emphasized the importance of having a common ground for the discussion of decision processes based on a theoretical and scientific framework. The framework should be based on generally agreed principles. We suggest the common ground be based on the mathematics and empirical basis of physical theories as well as the foundations of ***game theory*** and related works in economics. In *decision process theory,* we take a unified and consistent view, which has led us to replacing static theories and equilibrium frameworks with the physical *principle of least action* articulated as the field equations, Eq. (3.1). We have demonstrated that this mathematical expression, though compact, in addition to providing the self-consistent framework summarizes a great deal of information that we believe is commonly understood to be part of the decision process. In section 3.4, using the *normal-form coordinate basis*, we expanded the field equations into three sets of equations and sketched the information that might be derived from each set.

To make further progress in extracting predictions from *decision process theory*, we consider models that have attributes in common with static models; this is in the same sense that DC circuits provide insight into AC circuits. We can use the same mathematics as long as we extend the interpretation of what the mathematics represents.

We don't however make those assumptions at the outset. Our bottom up approach is to proceed with the program set out in section 3.9 to find solvable models based on simplifications that are constructed to help us understand the features that are particular to *decision process theory*. We explore the properties of the *player fixed frame model* introduced in the last chapter, which highlight the role of agents and players as manifested by their ***vorticity, gradient*** and ***persistency*** properties. This is a general class of models that articulates large scale structure that we think will be attributes of every decision process. We emphasize that this approach is designed to help us understand the content of the theory.

We want to impose no practical barriers to considering small or large number of players, to including competition or to include variations depending on whether or not the games are fair. Furthermore, at the outset we want to consider that active decision flows are dynamic, which is to say that they change in time. We have to deal with some challenges however due to the complexity of the equations. The equations become increasingly more complex as the number of *active* strategies increases. These considerations have influenced our choice of assumptions with the *player fixed frame model* in section 3.8. Without limiting the number of active strategies, we impose the restriction that in the co-moving frame, the active strategy preferences are holonomic.

We find that the field equations and frame equations can be solved more easily than the wave equations that lead us back to the *harmonic coordinates*. We therefore break out some of the complexity of the problem by moving to the *co-moving orthonormal coordinate basis*. We think the field equations in the *co-moving frame* provide significant initial insight. Implicit in our move to the *co-moving frame* is an emphasis on those solutions that have *stationary* aspects. It will turn out that we have imposed the condition that there is a time-like isometry vector, which means we have the analogs of *gravity* and *gravitomagnetism*.

For the reader with some knowledge of theories of this type, we acknowledge that we are by no means the first to study Einstein type equations. We agree with the accumulated and accepted wisdom that the study of the field equations Eq. (3.1) is a large project that may take considerable effort and time to flesh out. Much work has been done already. However, our applications are different from those in physics and so the program that has been carried out in that domain is not entirely applicable here. Indeed without some analysis such as we suggest here, it is not clear what aspects can be usefully carried over.

It is important therefore to specify a program in our domain that identifies what is relatively simple from what is relatively complex. We start that project. Our goal is to at least get to the point that we can that it is possible to have a complete dynamical numerical solution for problems in which there are any number of strategies and any number of players. We should be able to carry out this program for one, two, three or four active strategies. As numerical techniques improve, it should be possible to extend to a larger number of strategies: such approaches have been used in other fields of study.

4.1.1 Player Fixed Frame Model—dynamic components

The key idea behind the ***player fixed frame model*** from section 3.8 is that in the co-moving frame of reference, the active strategy variables are exact, which imposes dynamic constraints. We made additional dynamic assumptions Eq. (3.50) to those of exactness, Eq. (3.46) and (3.49). We did not motivate those assumptions, a task we take up here:

$$\begin{aligned} & q^{\alpha} = 0 \\ & \omega_{\alpha\beta\upsilon} = \theta_{\alpha\beta} = \omega_{\alpha\beta\gamma} = 0 \\ & \phi_{\alpha\beta} = -\omega_{\alpha\beta} = 0 \end{aligned} \qquad (4.1).$$

The first equation makes the plausible statement that the energy flow has no acceleration along the proper-inactive directions. The player exercises some persistency in his or her decision process.

In the second and third set of equations the interpretation is that the co-moving frame has a fixed orientation with respect to the players and that the only compression will be along the active strategy directions and will be captured by $\omega_{\upsilon\alpha\beta}$. In particular, all compression or expansion coefficients such as $\theta_{\alpha\beta}$ and the symmetric part of $\omega_{\alpha\beta\gamma} = \omega_{\alpha\gamma\beta}$ are zero. Along the flow direction, the frame rotation $\phi_{\alpha\beta}$ is zero and so the Fermi derivative of the frame rotation also vanishes in Eq. (3.44).

Though we explore the distinctions that arise from these concepts in more detail in section 4.4, here we give some initial insights here about additional frame rotations. First, we extend the notion of the Fermi derivative Eq. (3.45), which was defined relative to the flow and provided physical insight into the rotational behavior of the frame along the flow. We take the directions to be along the orthonormal directions and below, indicate with the Greek letter α both the component index and the scalar variable. In this paragraph, we temporarily suspend our notation from the last chapter and allow this Greek letter to represent both proper-active and proper-inactive strategies:

$$\begin{aligned} & \frac{D_F X_\mu}{\partial\alpha} = \frac{DX_\mu}{\partial\alpha} + \frac{DV^\nu}{\partial\alpha} X_\nu V_\mu - V^\nu X_\nu \frac{DV_\mu}{\partial\alpha} \\ & \frac{DX_\mu}{\partial\alpha} \equiv X_{\mu;\nu} E^\nu{}_\alpha \end{aligned} \qquad (4.2)$$

We express these Fermi derivatives of the orthonormal vector fields $\{E_{\mu\alpha}, E_{\mu\upsilon}\}$, using the expansions Eq. (3.41) and (3.43) for the orientation potentials:

$$\begin{aligned} & \frac{D_F E_{\mu\beta}}{\partial\alpha} = -\omega_{\beta\gamma\alpha} E_\mu{}^\gamma \\ & \frac{D_F V_\mu}{\partial\alpha} = 0 \end{aligned} \qquad (4.3)$$

Not only is the Fermi derivative of the flow zero along the direction of motion, it is zero along any of the orthonormal coordinates. The Fermi derivative of any frame component along any other frame direction is a rotation set by the orientation potential restricted to the transverse space.

We return now to our notation from the last chapter and state the consequence of our model assumptions Eq. (4.1) for the variation of $E_{j\beta}$:

$$\begin{aligned}\frac{D_F E_{j\beta}}{\partial\alpha} &= -\omega_{\beta\gamma\alpha}E_j^{\ \gamma} - \omega_{\beta\upsilon\alpha}E_j^{\ \upsilon} = 0\\ \frac{D_F E_{j\beta}}{\partial\upsilon} &= -\omega_{\beta\gamma\upsilon}E_j^{\ \gamma} - \omega_{\beta\upsilon'\upsilon}E_j^{\ \upsilon'} = 0\end{aligned} \tag{4.4}$$

Based on the *player fixed frame model*, the orthonormal vectors for the inactive space (the Killing vectors) remain fixed when moved along any of the transverse directions. In particular we see the value of having $\omega_{\alpha\beta\upsilon} = 0$ and $\omega_{\alpha\beta\gamma} = 0$.

We have already commented about the frame rotation $\phi_{\alpha\beta}$ being zero along the flow direction. Our assumptions amount to the requirement that the frames do not rotate along the flow or along any of the transverse orthonormal axes. Our assumptions can be summarized as saying that with this hypothesis, the co-moving orthonormal coordinate basis is a ***player fixed frame model***, justifying its name. The orientation of the proper player does not change. This class of models enhances the notion of player persistency and extends the notion of persistency to be an attribute that applies to the large scale structure. We do expect to see some aspects of game theory in our solutions. We will also see significant differences. In particular we see strong vorticity effects evidenced in the ***vorticity tensor*** components $\omega_{\upsilon\alpha} \quad \omega_{\upsilon\upsilon'}$.

4.1.2 Player fixed frame model—persistent components

Persistency is a consequence of the Killing equations, Eq. (3.62), which we summarize in Table 4-1. These results were derived in the last chapter and provide the essential results for the frame evolutions. For further details see section 3.9. In Table 4-1 we provide equations for the transformations that take us from the *normal-form coordinate basis* to the *co-moving orthonormal coordinate basis*. One of the goals is to obtain equations for the transformations from the *normal-form coordinate basis* to the *orthonormal coordinate basis*.

It is noteworthy that the inactive Killing vectors $\{E_j^{\ o} \quad E_j^{\ \alpha} \quad E_j^{\ \upsilon}\}$ are stationary for the *player fixed* proper time through a wave equation, assuming we can define a proper time variable. We see that the transformations from active to inactive involve the proper charge e_α, which is stationary. The Killing conditions provide sufficient gradient equations for each stationary field so that these fields are determined, given appropriate initial conditions.

Table 4-1: player fixed frame model—persistent and dynamic components

Distinction	Variable	Properties
Energy Flow	$E_a^{\ o}$ $E_j^{\ o}$	The active flow are not in general *stationary* and satisfy Eq. (3.57)based on the *exact* character of the active flow and on the wave equation: $g^{ab}\partial_a E^o_{\ b} = 0$
Proper-active strategy	$E_a^{\ \upsilon}$ $E_j^{\ \upsilon} = 0$	The wave equation Eq. (3.57) determines the proper active strategy in terms of the *harmonic coordinates*: $g^{ab}\partial_a\partial_b y^\upsilon = -q^\upsilon - \omega^{\upsilon\beta}_{\ \ \beta}$.
Active strategy	$E^a_{\ o}$ $E^a_{\ \alpha} = e_\alpha E^a_{\ o}$ $E^a_{\ \upsilon}$	We have the exactness conditions Eq. (3.60) for x^a and the *harmonic gauge condition* Eq. (3.71) $\Delta_o E^a_{\ \upsilon} - \Delta_\upsilon E^a_{\ o} = -q_\upsilon E^a_{\ o} - 2\theta_\upsilon^{\ \alpha} e_\alpha E^a_{\ o}$ $\Delta_\upsilon E^a_{\ \upsilon'} - \Delta_{\upsilon'} E^a_{\ \upsilon} = \left(-2\omega_{\upsilon'\upsilon} + 2\omega^\alpha_{\ \upsilon'\upsilon}e_\alpha\right)E^a_{\ o}$. $\left(1 + e_\alpha e^\alpha\right)\Delta_o E^a_{\ o} = -\Delta^\upsilon E^a_{\ \upsilon} + \left(q^\upsilon + \omega^{\upsilon\alpha}_{\ \ \alpha}\right)E^a_{\ \upsilon}$

Inactive strategy ***Killing vector*** ***Player potential***	$E_{j\alpha}$ E_{jo} $E_{j\upsilon} = E^j{}_\upsilon = 0$ $E^j{}_o$ $E^j{}_\alpha$	The proper time variations Eq. (3.65) $\Delta_o E_{j\alpha} = 0$ and Eq. (3.68) $\Delta_o E_j{}^o = 0$, are zero. The gradients are Eq. (3.67) and (3.64); the Killing vectors determine the payoffs Eq. (3.30) using Eq. (3.74): $\Delta_\upsilon E_{j\alpha} = -2\theta_{\upsilon\alpha} E_{jo} - \omega_{\upsilon\alpha}{}^\beta E_{j\beta}, \quad \Delta_\upsilon E_j{}^o = -q_\upsilon E_j{}^o - 2\omega_\upsilon{}^\alpha E_{j\alpha}$ $f^j{}_{o\alpha} = f^j{}_{\alpha\beta} = 0, \quad f^j{}_{\upsilon\upsilon'} = 2\omega_{\alpha\upsilon\upsilon'} E^{j\alpha} - 2\omega_{\upsilon\upsilon'} E^{jo}, \quad f^j{}_{\alpha\upsilon} = e_\alpha f^j{}_{o\upsilon}$. $f^j{}_{o\upsilon} = 2\left(q_\upsilon E^{jo} + \left(\omega_{\upsilon\alpha} + \theta_{\upsilon\alpha}\right)\left(E^{j\alpha} + e^\alpha E^{jo}\right) + \omega_{\upsilon\alpha\beta} e^\alpha E^{j\beta}\right)\left(1 - E^k{}_o E_k{}^o\right)$
proper charge	$E^a{}_\alpha = e_\alpha E^a{}_o$	Orthogonality leads to Eq. (3.54), (3.55), (3.59) and (3.69), showing in particular that the proper charge e_α is conserved and the charge gradient determined. $E_{jo} = -E_j{}^\alpha e_\alpha$ $\Delta_\alpha = e_\alpha \Delta_o$ $\Delta_o e_\alpha = 0$ $\Delta_\upsilon e_\alpha = -q_\upsilon e_\alpha + 2\omega_{\upsilon\alpha} - 2\theta_{\upsilon\beta} e^\beta e_\alpha + \omega_{\upsilon\alpha\beta} e^\beta$

4.2 *Field equations—co-moving orthonormal coordinate basis*

We gain insight about decision processes by working in the *co-moving frame*. The field equations Eq. (3.1) in the *co-moving orthonormal coordinate basis* are computed using the flux 2-form Eq. (2.27). The exercises from the last chapter addressed the technical aspects of this computation for the *normal-form coordinate basis*. In this chapter we outline the steps for the same calculation in the *co-moving basis*.

Our approach will be to take the expression for the orientation flux in term of the potentials and expand them. We provide a guide for all the time evolution equations in Table 4-2. We list all the space evolution equations in Table 4-3. The calculations in this section are for any model. In the next section we specialize to the *player fixed frame model*. By working through the calculations here and in the next section, along with the corresponding exercises at the end of the chapter, the reader will gain a deeper understanding of the field equations and their consequences. In section 4.4, these results will be applied to the identification of distinctions in the *player fixed frame model*.

In this section, we find it convenient to use more compact expressions, so we restrict Greek letters to not be along the flow: $\alpha, \beta, \gamma, \cdots \neq o$. We start with the defining relationships for the orientation flux Eq. (2.27):

$$\begin{aligned} \mathbf{R}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} &= d\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} + \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} \wedge \boldsymbol{\omega}^{\boldsymbol{\beta}}{}_{\mathbf{o}} \\ \mathbf{R}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} &= d\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} \wedge \boldsymbol{\omega}^{\mathbf{o}}{}_{\boldsymbol{\beta}} + \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\gamma}} \wedge \boldsymbol{\omega}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\beta}} \\ &\alpha, \beta, \gamma \neq o \end{aligned} \tag{4.5}$$

For the appropriate orientation potentials we use Eq. (3.41) and (3.43):

$$\begin{aligned} \omega_{o\alpha o} &= -q_\alpha \\ \omega_{o\alpha\beta} &= -\theta_{\alpha\beta} - \omega_{\alpha\beta} \\ \omega_{\alpha\beta o} &= -\phi^\alpha{}_\beta \\ &\alpha, \beta \neq o \end{aligned} \tag{4.6}$$

The orientation potential 1-forms that result from these definitions are:

$$\begin{aligned} \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} &= \omega^\alpha{}_{oo} \mathbf{V} + \omega^\alpha{}_{o\beta} \mathbf{E}^{\boldsymbol{\beta}} = q^\alpha \mathbf{V} + \left(\theta^\alpha{}_\beta + \omega^\alpha{}_\beta\right) \mathbf{E}^{\boldsymbol{\beta}} \\ \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} &= \omega^\alpha{}_{\beta o} \mathbf{V} + \omega^\alpha{}_{\beta\gamma} \mathbf{E}^{\boldsymbol{\gamma}} = -\phi^\alpha{}_\beta \mathbf{V} + \omega^\alpha{}_{\beta\gamma} \mathbf{E}^{\boldsymbol{\gamma}} \end{aligned} \tag{4.7}$$

Note the suggestive notation $\mathbf{V} = \mathbf{E}^{\mathbf{o}}$ for the flow 1-form. To obtain the differentials of the orientation potentials we use the differentials of the frames based on the condition of no torsion Eq. (1.54):

$$d\mathbf{V} = -\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right)\mathbf{E}^{\beta} \wedge \mathbf{E}^{\gamma} - q_{\beta}\mathbf{E}^{\beta} \wedge \mathbf{V}$$
$$d\mathbf{E}^{\alpha} = \omega^{\alpha}{}_{\beta\gamma}\mathbf{E}^{\beta} \wedge \mathbf{E}^{\gamma} + \left(\theta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} + \phi^{\alpha}{}_{\beta}\right)\mathbf{V} \wedge \mathbf{E}^{\beta} \tag{4.8}$$
$$\alpha, \beta, \gamma \neq o$$

We have sufficient information to compute the curvature components in Eq. (3.1). We take the mixed components in Eq. (4.7) and compute in exercise 7 the curvature components $R_{\alpha o\beta\gamma}$ and $R_{\alpha oo\beta}$, Eq. (4.115). From these curvature tensors, we obtain a variety of results that are detailed in the subsections that follow.

4.2.1 Time evolution of vorticity and expansion

Based on the symmetry condition $R_{\alpha oo\beta} = R_{\beta oo\alpha}$, Eq. (2.70), we get our first result, the time evolution of the vorticity:

$$\Delta_o \omega_{\alpha\beta} = \tfrac{1}{2}\left(q_{\alpha\|\beta} - q_{\beta\|\alpha}\right) + \theta^{\gamma}{}_{\alpha}\omega_{\beta\gamma} - \omega_{\alpha\gamma}\theta^{\gamma}{}_{\beta} + \phi_{\alpha\gamma}\omega^{\gamma}{}_{\beta} - \omega_{\alpha\gamma}\phi^{\gamma}{}_{\beta} \tag{4.9}$$

The double bar notation for the components of the covariant derivative is a short hand for the terms that involve only the transverse components:

$$q_{\alpha|\beta} = \Delta_{\beta}q_{\alpha} - \omega^{\gamma}{}_{\alpha\beta}q_{\gamma} - \omega^{o}{}_{\alpha\beta}q_{o} = q_{\alpha\|\beta} - \omega^{o}{}_{\alpha\beta}q_{o} = q_{\alpha\|\beta} \tag{4.10}$$

In other words, this is the covariant derivative defined on the hypersurface orthogonal to the *proper time coordinate*. It makes some of the intermediate algebra simpler, though in this case it makes no difference.

Next, we look at the symmetric curvature tensor $R_{\alpha oo\beta}$ component Eq. (4.115), called the ***tidal force*** by (Hawking & Ellis, 1973), which determines the time evolution of the expansion that appears in the literature:

$$\Delta_o \theta_{\alpha\beta} = -\theta_{\alpha\gamma}\theta^{\gamma}{}_{\beta} - \omega_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + R_{\alpha oo\beta} - q_{\alpha}q_{\beta} + \tfrac{1}{2}\left(q_{\alpha|\beta} + q_{\beta|\alpha}\right) + \phi_{\alpha\gamma}\theta^{\gamma}{}_{\beta} + \phi_{\beta\gamma}\theta^{\gamma}{}_{\alpha} \tag{4.11}$$

We have used Eq. (4.9) to eliminate the time dependence of the vorticity. We return to Eq. (4.11) later, though we note that this is a useful result in its own right.

We obtain a field equation by contracting the tidal force using Eq. (2.74) with n space components that total the number of active and inactive strategies. We obtain an expression in which only the energy density and average pressure appears:

$$R_{oo} = R^{\alpha}{}_{oo\alpha} = \kappa\left(-\mu - p + \frac{\mu - p}{n-1}\right) \tag{4.12}$$

We combine this equation with the time evolution of the expansion to get the evolution of the volume compression coefficient $\theta = m^{\alpha\beta}\theta_{\alpha\beta}$:

$$\Delta_o \theta = q^{\alpha}{}_{|\alpha} - \theta^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \omega^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} + 2\phi^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \kappa\left(\mu + p - \frac{\mu - p}{n-1}\right) \tag{4.13}$$

This formula in the $(3+1)$ dimensions of physical space and time is attributed to Landau and Raychaudhuri by (Hawking & Ellis, 1973).

4.2.2 Codacci equations

Not all equations that we get are time evolution equations. The possibility of field equations being independent of time exists with Maxwell's equations for example: Coulomb's law provides a set of equations that are independent of time, with only spatial partial derivatives. For equations of the Einstein type, such relations are named after Codacci. In a space of n space dimensions and one time dimension, there is a theorem that there will be $n+1$ Codacci equations. We are able to identify n of these equations.

The mixed component $R_{o\beta}$ is $\kappa p_{o\beta}$, which is zero. This contracted curvature tensor is:

$$R_{o\beta} = R^{\gamma}{}_{o\beta\gamma} = m^{\alpha\gamma} R_{\alpha o\beta\gamma} \tag{4.14}$$

We use this with Eq. (4.115) from the exercises at the end of this chapter to obtain the n Codacci equations, (Hawking & Ellis, 1973, p. Eq. 7.17):

$$\theta^{\beta}{}_{\alpha\|\beta} + \omega^{\beta}{}_{\alpha\|\beta} - \theta_{\|\alpha} = -2\omega_{\alpha\beta} q^{\beta} \tag{4.15}$$

These constraints are on the surface orthogonal to the flow and contain no time derivatives. The constraints are only spatial partial differential equations that must be satisfied on the surface transverse to the time flow. Since there are $n+1$ Codacci relationships, we need to find one more.

4.2.3 Remaining equations

The remaining set of equations (line 3 and 5 of Table 4-2) are computed from the transverse components of the orientation potential 1-forms $\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$, Eq. (4.7). From exercise 8 at the end of the chapter, the differential 1-form Eq. (4.116) contains no proper time partial derivatives of the frame rotation $\phi_{\alpha\beta}$, suggesting that it is set to zero only by the Codacci constraints, or by our assumption of the *player fixed frame model*. However, it does not appear in the Codacci constraints Eq. (4.15), suggesting that there are not time evolution equations for the frame rotations $\phi_{\alpha\beta}$. We don't expect evolution equations for q_{α} or $\phi_{\alpha\beta}$, since their proper time derivatives don't occur in the curvature tensor components, though we may expect contributions from $\Delta_{\beta} q_{\upsilon} = e_{\beta} \Delta_{o} q_{\upsilon}$.

We do obtain proper time differential equations for the transverse orientation potentials $\omega_{\alpha\beta\gamma}$, Eq. (4.119) in exercise 11 as a result of the symmetry relations $R_{\alpha\beta o\gamma} = R_{\gamma o\beta\alpha}$ based on the computation of the curvature components Eq. (4.118). We have computed the time evolution of the vorticity, Eq. (4.9) and the time evolution of the transverse orientation potentials $\omega_{\alpha\beta\gamma}$, Eq. (4.119). We have yet to compute the evolution equations for the expansion parameters $\theta_{\alpha\beta}$ in order to have a complete set of evolution equations of the orientation potentials expressed in terms of the other potentials and the sources.

We obtain the complete equation for the expansion parameters by starting with the contracted curvature tensor in terms of the sources, Eq. (2.74):

$$R_{\alpha\beta} = R_{o\alpha\beta o} + R^{\gamma}{}_{\alpha\beta\gamma} = \kappa\left(p_{\alpha\beta} - p h_{\alpha\beta} + \frac{\mu - p}{n-1} h_{\alpha\beta} \right) \tag{4.16}$$

The result is based on the expression for the curvature tensor Eq. (4.115) and (4.118)

$$\begin{pmatrix} \Delta_o \theta_{\alpha\beta} + q_{\alpha} q_{\beta} - \tfrac{1}{2}\left(q_{\alpha\backslash\beta} + q_{\beta\backslash\alpha}\right) + \theta\theta_{\beta\alpha} - \phi_{\alpha\gamma}\theta^{\gamma}{}_{\beta} - \phi_{\beta\gamma}\theta^{\gamma}{}_{\alpha} \\ +\phi_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + \phi_{\beta\gamma}\omega^{\gamma}{}_{\alpha} + \omega^{\gamma}{}_{\alpha\varepsilon}\omega^{\varepsilon}{}_{\beta\gamma} - \tfrac{1}{2}\omega^{\gamma}{}_{\varepsilon\gamma}\omega^{\varepsilon}{}_{\alpha\beta} - \tfrac{1}{2}\omega^{\gamma}{}_{\varepsilon\gamma}\omega^{\varepsilon}{}_{\beta\alpha} \\ +\tfrac{1}{2}\Delta_{\alpha}\omega^{\gamma}{}_{\beta\gamma} + \tfrac{1}{2}\Delta_{\beta}\omega^{\gamma}{}_{\alpha\gamma} - \tfrac{1}{2}\Delta_{\gamma}\omega^{\gamma}{}_{\alpha\beta} - \tfrac{1}{2}\Delta_{\gamma}\omega^{\gamma}{}_{\beta\alpha} \end{pmatrix} = \kappa\left(p_{\alpha\beta} - p h_{\alpha\beta} + \frac{\mu - p}{n-1} h_{\alpha\beta} \right) \tag{4.17}$$

$$\alpha, \beta, \gamma, \cdots \neq o$$

The essential point is that these equations determine the time evolution of the expansion parameters entirely in terms of the sources and the orientation potentials and their derivatives along directions in the surface normal to the flow.

4.2.4 Summary and final Codacci equation

In summary, the complete set of evolution equations for the orientation potentials is:

$$\Delta_o\omega_{\alpha\beta} = \tfrac{1}{2}\left(q_{\alpha\|\beta} - q_{\beta\|\alpha}\right) + \theta^{\gamma}{}_{\alpha}\omega_{\beta\gamma} - \omega_{\alpha\gamma}\theta^{\gamma}{}_{\beta} + \phi_{\alpha\gamma}\omega^{\gamma}{}_{\beta} - \omega_{\alpha\gamma}\phi^{\gamma}{}_{\beta}$$

$$\Delta_o\omega_{\alpha\beta\gamma} = \begin{pmatrix} +\left(\theta_{\gamma\alpha} + \omega_{\gamma\alpha}\right)_{\|\beta} - \left(\theta_{\gamma\beta} + \omega_{\gamma\beta}\right)_{\|\alpha} + \omega^{\delta}{}_{\beta\gamma}\phi_{\alpha\delta} - \omega^{\delta}{}_{\alpha\gamma}\phi_{\beta\delta} - \omega_{\alpha\beta\delta}\left(\theta^{\delta}{}_{\gamma} + \omega^{\delta}{}_{\gamma} + b^{\delta}{}_{\gamma}\right) \\ -q_{\beta}\left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right) + q_{\alpha}\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) + 2q_{\gamma}\omega_{\alpha\beta} - \Delta_{\gamma}\phi_{\alpha\beta} + q_{\gamma}\phi_{\alpha\beta} \end{pmatrix}$$

$$\begin{pmatrix} \Delta_o\theta_{\alpha\beta} + q_{\alpha}q_{\beta} - \tfrac{1}{2}\left(q_{\alpha\|\beta} + q_{\beta\|\alpha}\right) + \theta\theta_{\beta\alpha} - \phi_{\alpha\gamma}\theta^{\gamma}{}_{\beta} - \phi_{\beta\gamma}\theta^{\gamma}{}_{\alpha} \\ +\phi_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + \phi_{\beta\gamma}\omega^{\gamma}{}_{\alpha} + \omega^{\gamma}{}_{\alpha\varepsilon}\omega^{\varepsilon}{}_{\beta\gamma} - \tfrac{1}{2}\omega^{\gamma}{}_{\varepsilon\gamma}\omega^{\varepsilon}{}_{\alpha\beta} - \tfrac{1}{2}\omega^{\gamma}{}_{\varepsilon\gamma}\omega^{\varepsilon}{}_{\beta\alpha} \\ +\tfrac{1}{2}\Delta_{\alpha}\omega^{\gamma}{}_{\beta\gamma} + \tfrac{1}{2}\Delta_{\beta}\omega^{\gamma}{}_{\alpha\gamma} - \tfrac{1}{2}\Delta_{\gamma}\omega^{\gamma}{}_{\alpha\beta} - \tfrac{1}{2}\Delta_{\gamma}\omega^{\gamma}{}_{\beta\alpha} \end{pmatrix} = \kappa\left(p_{\alpha\beta} - ph_{\alpha\beta} + \frac{\mu - p}{n-1}h_{\alpha\beta}\right) \qquad (4.18)$$

$$\Delta_o\theta = q^{\alpha}{}_{\|\alpha} - q^{\alpha}q_{\alpha} - \theta^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \omega^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} + 2\phi^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \kappa\left(\mu + p - \frac{\mu - p}{D-2}\right)$$

These are Einstein's field equations. Exercise 16 demonstrates that the last two equations are consistent, using Eq. (4.123). As a consequence we identify the remaining Codacci relation:

$$\Delta_{\alpha}\omega^{\gamma\alpha}{}_{\gamma} + \tfrac{1}{2}\theta^{2} - \tfrac{1}{2}\theta^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \tfrac{1}{2}\omega^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} + \phi^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} + \tfrac{1}{2}\omega^{\alpha\beta}{}_{\gamma}\omega^{\gamma}{}_{\beta\alpha} - \tfrac{1}{2}\omega^{\alpha}{}_{\beta\alpha}\omega^{\beta\gamma}{}_{\gamma} = \kappa\mu \qquad (4.19)$$
$$\alpha,\beta,\gamma \neq o$$

This equation shows explicitly that the energy density is the source of curvature since a non-zero value of the energy density requires non-zero values of at least some of the orientation potentials.

The exposition of the theory is complete in the *co-moving orthonormal coordinate basis*. We make the following observations, summarized in the table below for a space time of dimension $D = n+1$.

Table 4-2: Time Equations

Determination	Source	Number in dimension D	Number in dimension 4
q_{α}	Flow equation	$D-1$	3
$\phi_{\alpha\beta}$	Gauge choice (e.g. exactness)	$\tfrac{1}{2}(D-1)(D-2)$	3
$\Delta_o\theta_{\alpha\beta}$	Field Eq. (4.17)	$\tfrac{1}{2}D(D-1)$	6
$\Delta_o\omega_{\alpha\beta}$	Symmetry $R_{\alpha oo\beta} = R_{\beta oo\alpha}$ Eq. (4.9)	$\tfrac{1}{2}(D-1)(D-2)$	3
$\Delta_o\omega_{\alpha\beta\gamma}$	Symmetry $R_{\alpha\beta o\gamma} = R_{\gamma o\beta\alpha}$ Eq. (4.119)	$\tfrac{1}{2}(D-1)^{2}(D-2)$	9
Total equations	Orientation potentials	$\tfrac{1}{2}D^{2}(D-1)$	24

The other set of relationships are constraints, partial differential equations on the surface orthogonal to the direction of proper time and are listed below, Table 4-3. Structurally, the curvature tensor is antisymmetric in the first two indices and the last two indices, so the total number of components is $\tfrac{1}{4}D^{2}(D-1)^{2}$. The number of independent components is further reduced by the symmetry properties. The number of symmetry equations is $\tfrac{1}{8}D(D^{2}-1)(D-2)$, plus the Bianchi identities, $\tfrac{1}{24}D(D-1)(D-2)(D-3)$, which total $\tfrac{1}{6}D^{2}(D-1)(D-2)$. The net number of independent components is $\tfrac{1}{12}D^{2}(D^{2}-1)$ (*Cf.* section 2.9, exercise 13).

The total number of field equations is $\tfrac{1}{2}D(D+1)$. Of these, the time equations, Table 4-1, determine the compression matrix and the gradient equations, Table 4-2, determine the Codacci equations. In the *co-moving orthonormal frame*, the field equations are not sufficient to determine the *orientation (acceleration) potentials*; however they are the only equations that depend on the *inertial* properties. In this frame we need the symmetry equations to determine the time evolution of the remaining *orientation potentials*.

Table 4-3: Gradient Equations on Surface normal to flow

Determination	Source	Number for D dimensions	Number for 4 dimensions
μ	Field Eq. (4.19)	1	1
$R_{o\beta}=0$	Field Eq. (4.15)	$D-1$	3
$R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}$	Symmetry Eq. (4.121)	$\frac{1}{8}D(D-1)(D-2)(D-3)$	3
$R_{\alpha\beta\gamma\delta}+R_{\alpha\delta\beta\gamma}+R_{\alpha\gamma\delta\beta}=0$	Bianchi identity[16] Eq. (4.124)	$\frac{1}{24}(D-1)(D-2)(D-3)(D-4)$	0
$R_{o\alpha\beta\gamma}+R_{o\gamma\alpha\beta}+R_{o\beta\gamma\alpha}=0$	Bianchi identity[17] Eq. (4.124)	$\frac{1}{6}(D-1)(D-2)(D-3)$	1
Total spatial equations	Gradient equations		8

4.2.5 Conservation laws

We end this section with the conservation laws Eq. (2.42) in the *co-moving orthonormal frame*. These conservation laws are imposed on the sources because of the field equations, see the exercise 16 in section 2.9, Eq. (2.75). We have both a transverse and longitudinal part:

$$\begin{gathered}\left(\mu m^{\alpha\beta}+p^{\alpha\beta}\right)q_\beta=h^\alpha{}_\beta p^{\beta\gamma}{}_{|\delta}h^\delta{}_\gamma\\ \frac{d\mu}{d\tau}+\mu V^\alpha{}_{|\alpha}+V_{\alpha|\beta}p^{\alpha\beta}=0\end{gathered}\tag{4.20}$$

Transformed to the co-moving frame the first conservation law has no components along the flow direction: The acceleration is purely transverse, $\dot{V}_\alpha=q_\alpha$. The transverse conservation law along with the longitudinal conservation law can be written as (exercise 17):

$$\begin{gathered}\alpha,\beta,\gamma\neq o\\ \left(\mu m^\alpha{}_\beta+p^\alpha{}_\beta\right)q^\beta=\Delta_\beta p^{\alpha\beta}+\omega^\alpha{}_{\beta\gamma}p^{\beta\gamma}+\omega^\gamma{}_{\beta\gamma}p^{\alpha\beta}\\ \Delta_o\mu+\theta\mu+\theta_{\alpha\beta}p^{\alpha\beta}=0\end{gathered}\tag{4.21}$$

The first equation is algebraic and linear for q^β in which all the indices are transverse to the flow.

4.3 ***Field equations—player fixed frame model***

To fully articulate the consequences of the *player fixed frame model*, we apply all of the equations identified in section 4.2 and summarized in Table 4-2 and Table 4-3. We must in fact verify that the model assumptions are self-consistent as well as show that the equations have solutions. We do that in this section. These calculations provide deeper insight into our model assumption as well as further insight into the general theory. If questions arise about these results, their origins will be clear and subject to verification and challenge.

The results are summarized in Table 4-5 and Table 4-6. We provide exercises at the end of the chapter as guides for these results. Equally important are the distinctions about decisions that are the outcome of these calculations, which we apply to the orientation potentials. We discuss the distinctions in the next section. In this section, we revert to the notation in which Greek letters at the beginning of the alphabet $\alpha,\beta,\gamma,\cdots$ represent proper inactive strategies and the Greek letter upsilon, $\upsilon,\upsilon',\upsilon'',\cdots$ along with primes represent the proper active strategies.

[16] See (Gockeler & Schucker, 1987) and *Cf.* section 4.8, exercise 19.
[17] See (Gockeler & Schucker, 1987) and *Cf.* section 4.8, exercise 19.

4.3.1 Expansion equations

We start with the first set of equations in Table 4-2, the field equations Eq. (4.17) for $\{\theta_{\upsilon\upsilon'} \quad \theta_{\alpha\beta} \quad \theta_{\upsilon\alpha}\}$, the expansion coefficients. Based on the model assumptions, the expansion coefficients are zero in both the active and inactive space, but not necessarily the mixed. The time variations for the first two must be set to zero in Eq. (4.17) based on these model assumptions:

$$\begin{aligned}\Delta_o\theta_{\upsilon\upsilon'} &= 0\\ \Delta_o\theta_{\alpha\beta} &= 0\end{aligned} \tag{4.22}$$

When both components are both active, the first equation simplifies to:

$$\begin{pmatrix} -\tfrac{1}{2}\Delta_{\upsilon'}q_\upsilon - \tfrac{1}{2}\Delta_\upsilon q_{\upsilon'} - \tfrac{1}{2}\Delta_\upsilon\omega_{\upsilon'\gamma}{}^{\gamma} - \tfrac{1}{2}\Delta_{\upsilon'}\omega_{\upsilon\gamma}{}^{\gamma} + q_\upsilon q_{\upsilon'} \\ +2\phi_{\upsilon\alpha}\phi_{\upsilon'}{}^{\alpha} + 2\omega_\upsilon{}^{\upsilon''}\omega_{\upsilon'\upsilon''} + 2\omega_{\alpha\upsilon}{}^{\upsilon''}\omega^{\alpha}{}_{\upsilon'\upsilon''} + \omega^{\alpha}{}_{\upsilon\beta}\omega^{\beta}{}_{\upsilon'\alpha} \end{pmatrix} = \kappa\left(p_{\upsilon\upsilon'} - ph_{\upsilon\upsilon'} + \frac{\mu - p}{n-1}h_{\upsilon\upsilon'}\right) \tag{4.23}$$

$$\alpha, \beta, \cdots \neq o, \upsilon$$

This equation provides a partial differential equation for the acceleration of energy at each point in proper time.

When both components are inactive, the resultant Eq. (4.17) is a divergence condition for $\Delta_\upsilon\omega^{\upsilon}{}_{\alpha\beta}$:

$$\Delta_\upsilon\omega^{\upsilon}{}_{\alpha\beta} = \omega^{\upsilon}{}_{\alpha\beta}q_\upsilon + \omega_{\upsilon\gamma}{}^{\gamma}\omega^{\upsilon}{}_{\alpha\beta} - 2\theta_{\upsilon\alpha}\theta^{\upsilon}{}_{\beta} + 2\omega_{\upsilon\alpha}\omega^{\upsilon}{}_{\beta} + \omega^{\upsilon}{}_{\alpha\upsilon'}\omega^{\upsilon'}{}_{\beta\upsilon} - \kappa\left(p_{\alpha\beta} - ph_{\alpha\beta} + \frac{\mu - p}{n-1}h_{\alpha\beta}\right) \tag{4.24}$$

In general the divergence is not sufficient to determine the potentials.

4.3.2 Six symmetry classes

To determine the potentials $\omega^{\upsilon}{}_{\alpha\beta}$ we need the consistency equation $R_{\alpha\beta\upsilon\upsilon'} = R_{\upsilon\upsilon'\alpha\beta}$ from Table 4-3 in which two components are active and two inactive, Eq. (4.120), which simplifies to:

$$e_\alpha\Delta_o\omega_{\beta\upsilon\upsilon'} - e_\beta\Delta_o\omega_{\alpha\upsilon\upsilon'} + \omega_{\upsilon\alpha\gamma}\omega_{\upsilon'\beta}{}^{\gamma} - \omega_{\upsilon\beta\gamma}\omega_{\upsilon'\alpha}{}^{\gamma} + 2\omega_{\upsilon\alpha}\theta_{\upsilon'\beta} + 2\theta_{\upsilon\alpha}\omega_{\upsilon'\beta} - 2\omega_{\upsilon'\alpha}\theta_{\upsilon\beta} - 2\theta_{\upsilon'\alpha}\omega_{\upsilon\beta} = 0 \tag{4.25}$$

This will pose an interesting constraint on models as it involves the commutator of the strategic compression/rotation matrices. In addition there is a second consistency equation $R_{\alpha\upsilon\delta\upsilon'} = R_{\delta\upsilon'\alpha\upsilon}$ that follows from Eq. (4.120), which again after simplification is:

$$e_\alpha\Delta_o\omega_{\beta\upsilon\upsilon'} + e_\beta\Delta_o\omega_{\alpha\upsilon\upsilon'} - \Delta_\upsilon\omega_{\upsilon'\beta\alpha} + \Delta_{\upsilon'}\omega_{\upsilon\alpha\beta} - 2\omega_{\upsilon'\alpha}\theta_{\upsilon\beta} + 2\theta_{\upsilon'\alpha}\omega_{\upsilon\beta} - 2\theta_{\upsilon\alpha}\omega_{\upsilon'\beta} + 2\theta_{\upsilon'\beta}\omega_{\upsilon\alpha} = 0 \tag{4.26}$$

These two equations combine to form a single equation, which gives back each equation when extracting the symmetric and antisymmetric part in α, β:

$$2e_\alpha\Delta_o\omega_{\beta\upsilon\upsilon'} = \Delta_\upsilon\omega_{\upsilon'\beta\alpha} - \Delta_{\upsilon'}\omega_{\upsilon\alpha\beta} - \omega_{\upsilon\alpha\gamma}\omega_{\upsilon'\beta}{}^{\gamma} + \omega_{\upsilon\beta\gamma}\omega_{\upsilon'\alpha}{}^{\gamma} - 4\omega_{\upsilon\alpha}\theta_{\upsilon'\beta} + 4\omega_{\upsilon'\alpha}\theta_{\upsilon\beta} \tag{4.27}$$

If the time dependence is zero this equation determines the "curl" of $\omega_{\upsilon\alpha\beta}$. In general the divergence and curl are sufficient to determine the potentials (see exercise 18). The additional constraint on the matrices emphasizes that these potentials are matrices in the inactive space.

The results of the symmetries Eq. (4.120) from Table 4-3 were helpful in providing equations that determine the orientation potentials. We should consider all such relations. There are six classes of consistency equations involving active and inactive indices that follow from these symmetries:

$$\begin{aligned} R_{\alpha\beta\upsilon\upsilon'} &= R_{\upsilon\upsilon'\alpha\beta}\\ R_{\alpha\upsilon\delta\upsilon'} &= R_{\delta\upsilon'\alpha\upsilon}\\ R_{\alpha\beta\delta\gamma} &= R_{\delta\gamma\alpha\beta}\\ R_{\alpha\beta\delta\upsilon} &= R_{\delta\upsilon\alpha\beta}\\ R_{\alpha\upsilon\upsilon'\upsilon''} &= R_{\upsilon'\upsilon''\alpha\upsilon}\\ R_{\upsilon\upsilon'\upsilon''\upsilon'''} &= R_{\upsilon''\upsilon'''\upsilon\upsilon'} \end{aligned} \tag{4.28}$$

We have covered the first two. The third and sixth are identically satisfied in the *player fixed frame model*. The fourth is:

$$e_\alpha \Delta_o \omega_{\upsilon\delta\beta} = e_\beta \Delta_o \omega_{\upsilon\delta\alpha} \tag{4.29}$$

The fifth gives the "curl" of the payoff in the co-moving basis, after the usual simplifications:

$$\begin{pmatrix} +\Delta_\upsilon \omega_{\alpha\upsilon'\upsilon''} - 2\theta_{\upsilon\alpha}\omega_{\upsilon'\upsilon''} - \omega_{\alpha\upsilon\beta}\omega^{\beta}{}_{\upsilon'\upsilon''} \\ +\Delta_{\upsilon''}\omega_{\alpha\upsilon\upsilon'} - 2\theta_{\upsilon''\alpha}\omega_{\upsilon\upsilon'} - \omega_{\alpha\upsilon''\beta}\omega^{\beta}{}_{\upsilon\upsilon'} \\ +\Delta_{\upsilon'}\omega_{\alpha\upsilon''\upsilon} - 2\theta_{\upsilon'\alpha}\omega_{\upsilon''\upsilon} - \omega_{\alpha\upsilon'\beta}\omega^{\beta}{}_{\upsilon''\upsilon} \end{pmatrix} = 0 \tag{4.30}$$

This "curl" is analogous to n magnetic monopoles in Eq. (1.20) expressed in covariant notation. In the holonomic frame, these conditions imply that the payoff field can be written in terms of a potential, Eq. (1.21).

4.3.3 Time evolution of the "mixed" expansion coefficient

To complete analysis of the time evolutions Eq. (4.17) for the *player fixed frame model*, we write the result, after simplifications, for the expansion coefficient $\theta_{\upsilon\alpha}$, which in general provides the time evolution:

$$\begin{pmatrix} \Delta_o \theta_{\alpha\upsilon} - \Delta_{\upsilon'}\omega_{\alpha\upsilon}{}^{\upsilon'} + \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} - 2\omega^{\upsilon'}{}_{\alpha}\omega_{\upsilon\upsilon'} + \omega^{\upsilon'}{}_{\alpha\beta}\omega^{\beta}{}_{\upsilon\upsilon'} + \omega_{\upsilon'\beta}{}^{\beta}\omega_{\alpha\upsilon}{}^{\upsilon'} \\ -\tfrac{1}{2}e_\alpha \Delta_o q_\upsilon - \tfrac{1}{2}e_\alpha \Delta_o \omega_{\upsilon\beta}{}^{\beta} + \tfrac{1}{2}e_\beta \Delta_o \omega_{\upsilon\alpha}{}^{\beta} \end{pmatrix} = \kappa p_{\alpha\upsilon} \tag{4.31}$$

If we think of the potential $\omega_{\alpha\upsilon\upsilon}$ as the payoff matrix for a ***proper-player*** α in the co-moving frame, it is analogous to the magnetic field in Maxwell's equations, Eq. (1.19). We can view Eq. (4.31) as the analog of Ampère's law suggesting that the expansion coefficient $\theta_{\upsilon\alpha}$ is analogous to the electric field and its contribution in Eq. (4.31) is the analog of the displacement current in electrical engineering[18]. We then view $\kappa p_\alpha{}^\upsilon$ as the current along the strategic direction υ for proper-player α. We shall firm up these distinctions in the next section.

4.3.4 Codacci equations

We tackle item two next in Table 4-3, which provide the consequences of the field equations that result from $R_{o\beta} = 0$ that we have named the Codacci equations, Eq. (4.15). After simplification, in the *player fixed frame model*, the proper inactive components give the divergence of the mixed sum of expansion and vorticity components:

$$\Delta_\upsilon \left(\theta^{\upsilon}{}_{\alpha} + \omega^{\upsilon}{}_{\alpha}\right) = \omega^{\upsilon\upsilon'}\omega_{\alpha\upsilon\upsilon'} + 2\omega_{\upsilon\alpha}q^{\upsilon} + \omega^{\upsilon}{}_{\alpha\beta}\left(\theta_{\upsilon}{}^{\beta} - \omega_{\upsilon}{}^{\beta}\right) + \omega^{\upsilon\beta}{}_{\beta}\left(\theta_{\upsilon\alpha} + \omega_{\upsilon\alpha}\right) \tag{4.32}$$

The active component gives the divergence of the active vorticity components:

$$\Delta_{\upsilon'}\omega^{\upsilon'}{}_{\upsilon} = -\omega_{\upsilon\upsilon'}\left(\omega^{\upsilon'\alpha}{}_{\alpha} + 2q^{\upsilon'}\right) + 2\theta^{\upsilon'}{}_{\alpha}\omega^{\alpha}{}_{\upsilon\upsilon'} - e^{\alpha}\Delta_o\left(\theta_{\upsilon\alpha} - \omega_{\upsilon\alpha}\right) \tag{4.33}$$

We have a total of n such relations including both the active and inactive components. The inactive set Eq. (4.32) consists of terms reminiscent of Coulomb's law Eq. (1.14), though we are missing the charge and have the extra divergence $\Delta_\upsilon \omega^{\upsilon}{}_{\alpha}$.

To complete the items in Table 4-3 we need only do the first element Eq. (4.19), which after model simplification gives the energy density in terms of the potentials:

$$\kappa\mu = \begin{pmatrix} -\Delta_\upsilon \omega^{\upsilon\alpha}{}_{\alpha} - \theta_{\upsilon}{}^{\alpha}\theta^{\upsilon}{}_{\alpha} + 3\omega_{\upsilon}{}^{\alpha}\omega^{\upsilon}{}_{\alpha} + 2\theta_{\upsilon}{}^{\alpha}\omega^{\upsilon}{}_{\alpha} + \tfrac{3}{2}\omega^{\upsilon}{}_{\upsilon'}\omega_{\upsilon}{}^{\upsilon'} \\ +\tfrac{1}{2}\omega_{\alpha}{}^{\upsilon\upsilon'}\omega^{\alpha}{}_{\upsilon\upsilon'} + \tfrac{1}{2}\omega^{\upsilon\alpha}{}_{\beta}\omega_{\upsilon\alpha}{}^{\beta} + \tfrac{1}{2}\omega_{\upsilon\alpha}{}^{\alpha}\omega^{\upsilon\beta}{}_{\beta} \end{pmatrix} \tag{4.34}$$

---

[18] Electric and magnetic fields depend on the frame of reference. Our terminology in this section is specific to the *co-moving frame of reference*.

We already have an expression for the divergence, Eq. (4.24), so an alternate expression is one in which the partial derivatives are absent:

$$\begin{pmatrix} \kappa\left(p^{\alpha}{}_{\alpha} - ph^{\alpha}{}_{\alpha} - \mu + \dfrac{\mu - p}{n-1} h^{\alpha}{}_{\alpha} \right) + \theta_{\upsilon}{}^{\alpha}\theta^{\upsilon}{}_{\alpha} + 2\theta^{\upsilon}{}_{\alpha}\omega_{\upsilon}{}^{\alpha} + \omega_{\upsilon\alpha}\omega^{\upsilon\alpha} \\ -\omega^{\upsilon\beta}{}_{\beta} q_{\upsilon} + \tfrac{3}{2}\omega^{\upsilon}{}_{\upsilon'}\omega_{\upsilon}{}^{\upsilon'} + \tfrac{3}{2}\omega^{\alpha}{}_{\upsilon\upsilon'}\omega_{\alpha}{}^{\upsilon\upsilon'} + \tfrac{1}{2}\omega^{\upsilon\alpha}{}_{\beta}\omega_{\upsilon\alpha}{}^{\beta} - \tfrac{1}{2}\omega_{\upsilon\alpha}{}^{\alpha}\omega^{\upsilon\beta}{}_{\beta} \end{pmatrix} = 0 \qquad (4.35)$$

We have an algebraic set of relations between the energy density and diagonal stress components with the orientation potentials. This completes the analysis of the items in Table 4-3.

4.3.5 Vorticity equations

We turn to row four of the time evolutions, Table 4-2, which gives the time evolution of the vorticity components, Eq. (4.9) that result from the symmetry $R_{\alpha o o \beta} = R_{\beta o o \alpha}$. We write out these conditions for the various cases, $\{\omega_{\upsilon\upsilon'} \quad \omega_{\upsilon\alpha} \quad \omega_{\alpha\beta}\}$, starting with the first case that both indices are active. After simplification the result for the *player fixed frame model* is:

$$\Delta_o \omega_{\upsilon\upsilon'} = \tfrac{1}{2}\left(\Delta_{\upsilon'} q_{\upsilon} - \Delta_{\upsilon} q_{\upsilon'}\right) + 2\theta_{\upsilon}{}^{\alpha}\omega_{\upsilon'\alpha} - 2\omega_{\upsilon\alpha}\theta_{\upsilon'}{}^{\alpha} \qquad (4.36)$$

The second case is the vorticity with mixed components, whose simplified expression is:

$$\Delta_o \omega_{\upsilon\alpha} = \tfrac{1}{2} e_{\alpha} \Delta_o q_{\upsilon} \qquad (4.37)$$

The third case gives the time evolution for the inactive components, which after simplification is:

$$\Delta_o \omega_{\alpha\beta} = 0 \qquad (4.38)$$

We set these vorticity components to zero on the initial hypersurface and this shows that they remain zero at all later proper times along an appropriate path.

4.3.6 Remaining potentials and their evolution

The last orientation potentials to analyze from Table 4-2 are the time evolutions of the last row. There are six cases to consider:

$$\omega_{\upsilon\alpha\beta} \quad \omega_{\alpha\upsilon\upsilon'} \quad \omega_{\upsilon\upsilon'\upsilon''} \quad \omega_{\alpha\beta\upsilon} \quad \omega_{\alpha\beta\gamma} \quad \omega_{\upsilon\upsilon'\alpha} \qquad (4.39)$$

We start with the first case $\omega_{\upsilon\alpha\beta}$ and simplify Eq. (4.119) using the model assumptions:

$$\Delta_o \omega_{\upsilon\alpha\beta} = e_{\alpha}\Delta_o\left(\theta_{\upsilon\beta} - \omega_{\upsilon\beta}\right) + e_{\beta}\Delta_o\left(\theta_{\upsilon\alpha} + \omega_{\upsilon\alpha}\right) - \omega_{\upsilon\upsilon'}\left(\omega^{\upsilon'}{}_{\alpha\beta} - \omega^{\upsilon'}{}_{\beta\alpha}\right) \qquad (4.40)$$

In the model, we have the symmetry $\omega_{\upsilon\alpha\beta} = \omega_{\upsilon\beta\alpha}$. The antisymmetric part of this equation is:

$$\Delta_o\left(\omega_{\upsilon\alpha\beta} - \omega_{\upsilon\beta\alpha}\right) = -2\omega_{\upsilon\upsilon'}\left(\omega^{\upsilon'}{}_{\alpha\beta} - \omega^{\upsilon'}{}_{\beta\alpha}\right) \qquad (4.41)$$

If this starts at zero it stays at zero over time. The symmetric part of Eq. (4.40) is:

$$\Delta_o \omega_{\upsilon\alpha\beta} = e_{\alpha}\Delta_o\theta_{\upsilon\beta} + e_{\beta}\Delta_o\theta_{\upsilon\alpha} \qquad (4.42)$$

This assumes that the antisymmetric part of $\omega_{\upsilon\alpha\beta}$ is in fact zero. It then follows from this and Eq. (4.29) that:

$$e_{\alpha}\Delta_o\theta_{\upsilon\beta} = e_{\beta}\Delta_o\theta_{\upsilon\alpha} \qquad (4.43)$$

This allows for a simplification of the analog of Ampère's Law, Eq. (4.31):

$$\Delta_o\theta_{\alpha\upsilon} = \Delta_{\upsilon'}\omega_{\alpha\upsilon}{}^{\upsilon'} - \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} + 2\omega^{\upsilon'}{}_{\alpha}\omega_{\upsilon\upsilon'} - \omega^{\upsilon'}{}_{\alpha\beta}\omega^{\beta}{}_{\upsilon\upsilon'} - \omega_{\upsilon'\beta}{}^{\beta}\omega_{\alpha\upsilon}{}^{\upsilon'} + \tfrac{1}{2}e_{\alpha}\Delta_o q_{\upsilon} + \kappa p_{\alpha\upsilon} \qquad (4.44)$$

Sticking with the same elements of Table 4-2, we consider the second potential in Eq. (4.39). In the model, after simplification it yields:

$$\Delta_o \omega_{\alpha \upsilon \upsilon'} = \begin{pmatrix} -e_\alpha \Delta_o \omega_{\upsilon' \upsilon} + \Delta_\upsilon \left(\theta_{\upsilon' \alpha} + \omega_{\upsilon' \alpha} \right) - \Delta_{\upsilon'} \left(\theta_{\upsilon \alpha} + \omega_{\upsilon \alpha} \right) \\ -\omega_{\upsilon' \alpha}{}^{\beta} \left(\theta_{\upsilon \beta} - \omega_{\upsilon \beta} \right) + \omega_{\upsilon \alpha \beta} \left(\theta_{\upsilon'}{}^{\beta} - \omega_{\upsilon'}{}^{\beta} \right) \\ -q_\upsilon \left(\theta_{\upsilon' \alpha} - \omega_{\upsilon' \alpha} \right) + q_{\upsilon'} \left(\theta_{\upsilon \alpha} - \omega_{\upsilon \alpha} \right) \end{pmatrix} \tag{4.45}$$

The equations are antisymmetric in the active indices demonstrating consistency with the model assumptions Eq. (3.49).

The third case in Eq. (4.39) with all three indices active yields a non-zero time evolution, which after model simplification is set to zero based on the exactness constraints Eq. (3.49):

$$\Delta_o \omega_{\upsilon \upsilon' \upsilon''} = \begin{pmatrix} +\Delta_\upsilon \omega_{\upsilon' \upsilon''} - 2\omega_{\upsilon \alpha} \omega^{\alpha}{}_{\upsilon' \upsilon''} + q_\upsilon \omega_{\upsilon' \upsilon''} \\ +\Delta_{\upsilon'} \omega_{\upsilon'' \upsilon} - 2\omega_{\upsilon' \alpha} \omega^{\alpha}{}_{\upsilon'' \upsilon} + q_{\upsilon'} \omega_{\upsilon'' \upsilon} \\ +\Delta_{\upsilon''} \omega_{\upsilon \upsilon'} - 2\omega_{\upsilon'' \alpha} \omega^{\alpha}{}_{\upsilon \upsilon'} + q_{\upsilon''} \omega_{\upsilon \upsilon'} \end{pmatrix} = 0 \tag{4.46}$$

We obtain a set of "curl" equations for the vorticity potentials $\omega_{\upsilon \upsilon'}$.

The fourth case, $\omega_{\alpha \beta \upsilon}$ in Eq. (4.39), simplifies to:

$$\Delta_o \omega_{\alpha \beta \upsilon} = -e_\alpha \Delta_o \theta_{\upsilon \beta} + e_\beta \Delta_o \theta_{\alpha \upsilon} - e_\alpha \Delta_o \omega_{\upsilon \beta} + e_\beta \Delta_o \omega_{\upsilon \alpha} = 0 \tag{4.47}$$

It is identically zero (*i.e.* no new constraint) using Eq. (4.43) and (4.37). The fifth term $\omega_{\alpha \beta \gamma}$ computes to an identically zero time evolution:

$$\Delta_o \omega_{\alpha \beta \gamma} = 0 \tag{4.48}$$

The sixth term $\omega_{\upsilon \upsilon' \alpha}$ we compare with the time evolution of $\omega_{\upsilon \alpha \upsilon'} = -\omega_{\alpha \upsilon \upsilon'}$ Eq. (4.45) and take the difference, since the two are equal by the constraint Eq. (3.49). We make simplifications, obtaining the result:

$$\Delta_o \omega_{\upsilon \upsilon' \alpha} - \Delta_o \omega_{\upsilon \alpha \upsilon'} = 2 \begin{pmatrix} e_\alpha \Delta_o \omega_{\upsilon \upsilon'} + \Delta_\upsilon \omega_{\upsilon' \alpha} + q_\upsilon \omega_{\upsilon' \alpha} + \omega_{\upsilon \beta} \omega_{\upsilon' \alpha}{}^{\beta} \\ -\Delta_{\upsilon'} \omega_{\upsilon \alpha} - q_{\upsilon'} \omega_{\upsilon \alpha} - \omega_{\upsilon' \beta} \omega_{\upsilon \alpha}{}^{\beta} \end{pmatrix} = 0 \tag{4.49}$$

In this case the result is not automatically zero, so that we obtain the "curl" equation for $\omega_{\upsilon \alpha}$. Coupling this equation with the time variation Eq. (4.45), we achieve the following simpler expression:

$$\Delta_o \omega_{\alpha \upsilon \upsilon'} = \Delta_\upsilon \theta_{\upsilon' \alpha} - \Delta_{\upsilon'} \theta_{\upsilon \alpha} + q_{\upsilon'} \theta_{\upsilon \alpha} - q_\upsilon \theta_{\upsilon' \alpha} + \omega_{\upsilon \alpha}{}^{\beta} \theta_{\upsilon' \beta} - \omega_{\upsilon' \alpha}{}^{\beta} \theta_{\upsilon \beta} \tag{4.50}$$

Comparing this with Maxwell's equations Eq. (1.14), these equations are the generalization of Faraday's Law to decision theory.

4.3.7 Conservation laws

We have now computed all of the items in Table 4-2; we have not explicitly checked Eq. (2.71), but leave to an exercise to check that they are satisfied. The remaining equations result from the conservation laws on the inertial sources, Eq. (4.21). The longitudinal conservation law in Eq. (4.21) simplifies for the *player fixed frame model*:

$$\Delta_o \mu = -2\theta_{\upsilon \beta} p^{\upsilon \beta} \tag{4.51}$$

In addition, there are two transverse conservation laws to consider, one for accelerations along α and the other along υ. After simplification the first of these is:

$$\Delta_\upsilon p^{\upsilon \alpha} = p^{\upsilon \alpha} q_\upsilon - e_\beta \Delta_o p^{\alpha \beta} + \omega_{\upsilon \beta}{}^{\alpha} p^{\upsilon \beta} + \omega_{\upsilon \beta}{}^{\beta} p^{\upsilon \alpha} \tag{4.52}$$

The second is:

$$\Delta_{\upsilon'} p^{\upsilon \upsilon'} = \mu q^{\upsilon} + p^{\upsilon \upsilon'} q_{\upsilon'} + \omega_{\upsilon' \alpha}{}^{\alpha} p^{\upsilon \upsilon'} - \omega^{\upsilon}{}_{\alpha \beta} p^{\alpha \beta} - 2\omega_{\alpha \upsilon'}{}^{\upsilon} p^{\upsilon' \alpha} - e_\alpha \Delta_o p^{\upsilon \alpha} \tag{4.53}$$

These are the differential equations for the stress components.

4.4 *Player fixed frame model—distinctions*

The results from the last section are summarized in the three tables, Table 4-4, Table 4-5 and Table 4-6 in this section. We discuss the distinctions that arise from these results. In thinking about the results from the previous sections, it is tempting to base distinctions solely on concepts that arise in game theory, economics or other decision processes. In engineering however we are familiar with the problem. There are many types of engineering problems in each of the engineering disciplines. Historically, there were specialized distinctions associated with each particular application. The power of the engineering discipline and the scientific principle in general is to see the underlying connections that span multiple applications. Thus engineering terminology is often based on the terminology of the underlying physical laws, which are abstract and mathematically based. This makes the concepts somewhat more objective.

We see some advantage for decision theory to adopt the same stance and rely more on the physics nomenclature than application specific distinctions. We see an advantage with distinctions that are more neutral as they help provide the common ground for discussion. However, we balance this stance with the understanding that there is an advantage to use names that appear to apply to the subject matter as they make the arguments clearer. Thus we feel justified that in introducing decision theory, we relied on foundational aspects of *game theory*, section 1.1. We believe it useful to discuss effects with examples, where we propose how our theory applies. In general, we indicate by bold italics distinctions that follow directly from our theoretical foundation and that should be generally applicable.

We start our discussion of the results of the sections 4.1 and 4.3 based on these caveats. We find that there are four classes of forces that determine decisions: electric forces, magnetic forces, tidal forces and stress or inertial forces due to external sources. In the co-moving frame we are able to isolate the global characteristics of these four classes.

These four forces contribute to the change in flow through the transverse conservation of energy and momentum, Eq. (2.42). We recall the form of this with no inertial effects, Eq. (1.75):

$$g_{ab}\frac{dV^b}{d\tau}+\varpi_{abc}V^bV^c-V_kF^k_{ab}V^b-\tfrac{1}{2}V^jV^k\partial_a\gamma_{jk}=0 \qquad (4.54)$$

This shows that the acceleration (the first term) is determined by the active orientation potentials ϖ_{abc}, the electromagnetic forces $V_kF^k_{ab}V^b$ and additional forces due to the charges and the gradient of the inactive metric. It is worth noting that this latter term can be written as

$$-\tfrac{1}{2}V^jV^k\partial_a\gamma_{jk}=\left(q_\upsilon\left(e_\alpha e^\alpha\right)^2-2\left(\omega_\upsilon{}^\alpha+\theta_\upsilon{}^\alpha\right)e_\alpha e_\beta e^\beta+\omega_{\upsilon\alpha\beta}e^\alpha e^\beta\right)\frac{E_a{}^\upsilon}{\left(1+e_\alpha e^\alpha\right)^2} \qquad (4.55)$$

This shows how the co-moving frame orientation potentials contribute to the acceleration. To these effects we add the inertial fields.

4.4.1 Distinctions—persistency

The origin of ***persistency*** is the local group invariance of a commutative group of isometries and leads to the concept of a ***player*** in our *decision process theory*. A possible global concept is based on the idea that the transformation from the *normal-form coordinate basis* to the *co-moving basis* doesn't rotate the axes that correspond to the players. This is made explicit in the ***player fixed frame model*** from section 3.8. We provide a concept that is a global property, corresponding to real world behaviors. This notion of player transcends the transactions over time. As players make choices and change their decisions, we are still able to distinguish different players. Yet players may change their payoff matrix and change the frequencies with which they pick strategies. The *player fixed frame model* provides a mathematical basis for this type of view. It is a model or approximation that the general *decision process theory* does not require. We investigate this model and see what consequences it has and whether reality conforms to this view. So this model is an inquiry into a particular form of persistency that is a global property rather than just a local property. The consequences of these ideas are summarized in Table 4-1 in which we identify four key distinctions:

1. Energy flow
2. Active strategy and proper active strategy
3. Inactive strategy, proper inactive strategy, Killing vector and player potential vector field
4. Charge, proper charge, charge gradient

We discuss each of these distinctions in turn.

We defined a formal unit vector field $\mathbf{V} = \mathbf{E}^{\mathbf{o}}$ to be the direction along which the energy of the system flows at each point. The ***energy flow*** is an important attribute of any physical system. What is the meaning of energy and energy flow for decisions? For decision processes we identified in section 1.1 the energy flow direction as characterizing the strategy chosen: the relative frequencies of the flow are the relative frequencies with which the players collectively make their choice.

We now identify additional attributes that are associated with the flow of energy. In game theory and many economic theories, decisions are transactions in which something of value is exchanged. Game theory for example rests on the idea of a utility that each player gives or receives in a transaction, summarized by the payoff matrix of values. In our *decision process theory*, we exchange energy. How that occurs depends on the nature of the forces and is dictated by the *principle of least action* (*Cf.* section 2.4).

Energy is convertible and described for all forces by their contribution to the energy momentum tensor. Because we assume the foundational aspects of game theory, our notion of energy and the game theory notion of utility are consistent. We start with the same notion of a payoff matrix including a notion of some initial *game value*. The payoff field contributes to the energy momentum field so the specified payoff values also specify the initial energy and momentum of the system.

The basic attributes of and equations for persistency in the *player fixed frame model* are specified in Table 4-1. The energy flow consists of the ***active strategy*** components $E_a^{\ o}$ that reflect the strategic frequency choices. In the co-moving frame the ***proper active strategy*** components are along the υ axes. We distinguish the ***inactive strategy*** flow components $E_j^{\ o}$ as those that reflect the basic coupling of the player to the decision process. We also call these basic components the ***charge of player*** j. The projection of the charge onto the co-moving direction α we call the ***proper charge*** of player α. In the co-moving frame the ***proper inactive strategy*** components are along the α axes. We see from Table 4-1 that the projection is $E_j^{\ o} = -E_j^{\ \alpha} e_\alpha$, so it makes sense to call e_α the proper charge of α.

We made the formal distinction that inactive strategies correspond to isometry transformations and hence provide the theoretical foundation for what we mean by a ***player***. We made this notion covariant by noting that the necessary and sufficient condition for an isometry transformation is the existence of a Killing vector field. We identify the various components of this field as $E_{jo} \quad E_{j\alpha} \quad E_{j\upsilon}$ in the co-moving basis. Based on the relations that must be satisfied in general for such vectors, we obtain the equations in Table 4-1.

Our notion of a player and the underlying isometry mechanism was introduced to create the payoff field $F^j_{\ ab}$. The results in Table 4-1 show that the linear combinations of the Killing vectors $E^j_{\ o} \quad E^j_{\ \alpha} \quad E^j_{\ \upsilon}$ are the ***player potential vector fields*** that define the payoffs Eq. (3.30):

$$\begin{aligned} &\alpha, \beta \neq o \\ &f^j_{\ \alpha\beta} = E^j_{\ \beta\backslash\alpha} - E^j_{\ \alpha\backslash\beta} = F^j_{\ ab} E^a_{\ \alpha} E^b_{\ \beta} \end{aligned} \tag{4.56}$$

Thus the equations that determine the transformations of the inactive flows also determine the persistency and payoff properties of the theory. In the co-moving basis, the payoffs are specified by orientation potentials in Table 4-1.

Below, we identify the payoffs with the electric and magnetic forces. With that identification, the ***charge*** or ***proper charge*** for a given player is then the strength of the coupling of sources to the field for that player in the normal-form or co-moving coordinate basis respectively. There will be an additional coupling specified by the ***player current*** (section 4.4.4). There is no correspondence of these notions in game theory since two payoffs that differ by an overall constant factor have the same strategic behaviors. We do note however that the idea of ***engagement*** is an important concept in the real world. In a game such as poker or a real world situation such as a world war, the size of the stakes impacts how people play the game. So we might think of charge in practical applications as *engagement*.

4.4.2 Distinctions—electromagnetic fields

The compelling reason to identify the payoffs with the electromagnetic field is the form of the equations: either Eq. (3.16) in the *normal-form coordinate basis* or Table 4-4 in the co-moving coordinate basis. The results in the co-moving basis are striking. Comparing these equations with Maxwell's Eq. (1.14), we identify the ***electric field*** for player α with $\theta_{\upsilon\alpha}$ and the ***magnetic field*** for player α with $\omega_{\alpha\upsilon\upsilon'}$. Though striking, the equations are not exactly the same. The Maxwell-type equations for decision theory contain corrections due to tidal forces discussed below and reflect the existence of multiple independent player fields.

These equations as well as all the other equations that we obtain in this theory are of three general types that we characterize as follows:

1. ***Time evolution equations***: for example we have $\Delta_o\theta_{\upsilon\alpha}$ on the left-hand-side of the equation and only field values or the spatial derivatives on the right-hand-side.
2. ***Divergence equations***: The spatial derivative is contracted with the tensor on the left-hand-side of the equation, for example $\Delta_\upsilon\theta^\upsilon{}_\alpha$ and on the right-hand-side there are only fields or other divergences if they have not been specified in the theory.
3. ***Curl equations***: for vector fields we will have antisymmetric tensor combinations on the left-hand-side such as $\Delta_\upsilon\theta_{\upsilon'\alpha}-\Delta_{\upsilon'}\theta_{\upsilon\alpha}$ and on the left only fields (or in this case time derivatives). For two dimensional tensor fields such as $\omega_{\alpha\upsilon\upsilon'}$ there will be an equation that determines the totally antisymmetric cyclic combination $\Delta_\upsilon\omega_{\alpha\upsilon'\upsilon''}+\Delta_{\upsilon''}\omega_{\alpha\upsilon\upsilon'}+\Delta_{\upsilon'}\omega_{\alpha\upsilon''\upsilon}$.

Table 4-4: Player fixed frame model—electromagnetic fields

Distinction	Variable	Properties
Electric field	$\theta_{\upsilon\alpha}$	Ampère's law Eq. (1.19) is generalized by Eq. (4.44), which gives the time evolution. Eq. (4.32) and (4.43) provide the generalization of Coulomb's law: $\Delta_o\theta_{\alpha\upsilon}=\begin{pmatrix}\Delta_{\upsilon'}\omega_{\alpha\upsilon}{}^{\upsilon'}-\omega_{\alpha\upsilon\upsilon'}q^{\upsilon'}+2\omega^{\upsilon'}{}_\alpha\omega_{\upsilon\upsilon'}-\omega^{\upsilon'}{}_{\alpha\beta}\omega^\beta{}_{\upsilon\upsilon'}-\omega_{\upsilon'\beta}{}^\beta\omega_{\alpha\upsilon}{}^{\upsilon'}\\+\tfrac{1}{2}e_\alpha\Delta_o q_\upsilon+\kappa p_{\alpha\upsilon}\end{pmatrix}$ $e_\alpha\Delta_o\theta_{\upsilon\beta}=e_\beta\Delta_o\theta_{\upsilon\alpha}$ $\Delta_\upsilon\theta^\upsilon{}_\alpha=-\Delta_\upsilon\omega^\upsilon{}_\alpha+\omega^{\upsilon\upsilon'}\omega_{\alpha\upsilon\upsilon'}+2\omega_{\upsilon\alpha}q^\upsilon+\omega^\upsilon{}_{\alpha\beta}\left(\theta_\upsilon{}^\beta-\omega_\upsilon{}^\beta\right)+\omega^{\upsilon\beta}{}_\beta\left(\theta_{\upsilon\alpha}+\omega_{\upsilon\alpha}\right)$
Magnetic field	$\omega_{\alpha\upsilon\upsilon'}$	Eq. (4.30) gives the "curl" of the co-moving magnetic field. The time evolution is Eq. (4.50) generalizes Faraday's law Eq. (1.14): $\begin{pmatrix}+\Delta_\upsilon\omega_{\alpha\upsilon'\upsilon''}-2\theta_{\upsilon\alpha}\omega_{\upsilon'\upsilon''}-\omega_{\alpha\upsilon\beta}\omega^\beta{}_{\upsilon'\upsilon''}\\+\Delta_{\upsilon''}\omega_{\alpha\upsilon\upsilon'}-2\theta_{\upsilon''\alpha}\omega_{\upsilon\upsilon'}-\omega_{\alpha\upsilon''\beta}\omega^\beta{}_{\upsilon\upsilon'}\\+\Delta_{\upsilon'}\omega_{\alpha\upsilon''\upsilon}-2\theta_{\upsilon'\alpha}\omega_{\upsilon''\upsilon}-\omega_{\alpha\upsilon'\beta}\omega^\beta{}_{\upsilon''\upsilon}\end{pmatrix}=0$ $\Delta_o\omega_{\alpha\upsilon\upsilon'}=\Delta_\upsilon\theta_{\upsilon'\alpha}-\Delta_{\upsilon'}\theta_{\upsilon\alpha}+q_{\upsilon'}\theta_{\upsilon\alpha}-q_\upsilon\theta_{\upsilon'\alpha}+\omega_{\upsilon\alpha}{}^\beta\theta_{\upsilon'\beta}-\omega_{\upsilon'\alpha}{}^\beta\theta_{\upsilon\beta}$

By analogy with electro-dynamic equations, we expect that if the form matches Maxwell's equations there will be well defined solutions. Typically one needs both the divergence and curl equations as well as, or in combination with, a time evolution equation. That obtains here.

The concepts of electric and magnetic fields are precise; we expect that the consequence of the equations to be as rich and complex as those in the field of electrical engineering. One can skim any text book to see the diversity of effects that such equations generate. For example, see the classic text on electro-dynamics (Jackson, 1963). The question is whether these consequences accurately describe the effects of decision making. Decisions are often framed in more colorful language. Because of the origins of the distinction *payoffs* from game theory, we anticipate that the magnetic field reflects self-interest behaviors, egotism, competition and conflict. We recall the use of electric field components in converting a game into normal form, for example using time as the hedge strategy in Eq. (1.12). Based on this, we anticipate that the electric field reflects altruism, bias and persuasion. Democratic institutions argue the need for strong self-interests working together to find the common good. Autocratic institutions demonstrate the power of coercion to create conformity and agreement. We see that the electric and magnetic forces fuel these opposite extremes. Rather than argue in favor of any of these positions, our *decision process theory* makes visible the specific mechanisms indicated in Table 4-4, which provide a more precise set of distinctions than used in the popular press.

4.4.3 Distinctions—tidal fields

Although the inactive frames don't rotate along a particular proper strategic direction υ, the size of the space can increase or decrease. The measure of this change is the component of the orientation potential $\omega_{\alpha\beta}$. As an example, the volume of an element of ordinary space in spherical coordinates is $r^2 \sin\theta dr d\theta d\phi$. Effectively as one increases the radius, the directions of the latitude θ and longitude ϕ don't rotate. However the differential volume increases by r^2 reflecting the fact that the space is getting flatter. We term effects that reflect the curvature of the space ***tidal***. Generically the orientation potentials in the co-moving frame can be distinguished as ***spin*** or ***strain*** depending on whether potentials are anti-symmetric, for example $\omega_{\upsilon\alpha\beta} = -\omega_{\upsilon\beta\alpha}$ or symmetric, for example $\omega_{\upsilon\alpha\beta} = \omega_{\upsilon\beta\alpha}$, respectively. For these examples, the spin components are assumed to be zero in the *player fixed frame model*. These symmetric ***tidal bond*** components have no analogy in Newtonian physics. Strictly speaking, they are not attributes of Newtonian gravity since they are tensor forces. They are attributes of Einstein's general theory of relativity and do explain small but measurable effects such as the deviation of Mercury's path around the Sun compared to Newtonian calculations.

Table 4-5: Player fixed frame model—tidal fields

Distinction	Variable	Properties
Tidal charge gradient	$\omega_{\upsilon\alpha}$	"Curl" Eq. (4.49) and time evolution Eq. (4.37), where the time dependence of the tidal rotation is determined by Eq. (4.36) $\Delta_\upsilon \omega_{\upsilon'\alpha} - \Delta_{\upsilon'} \omega_{\upsilon\alpha} = -q_\upsilon \omega_{\upsilon'\alpha} + q_{\upsilon'} \omega_{\upsilon\alpha} - \omega_{\upsilon\beta} \omega_{\upsilon'\alpha}{}^{\beta} + \omega_{\upsilon'\beta} \omega_{\upsilon\alpha}{}^{\beta} - e_\alpha \Delta_o \omega_{\upsilon\upsilon'}$ $\Delta_o \omega_{\upsilon\alpha} = \tfrac{1}{2} e_\alpha \Delta_o q_\upsilon$
Tidal bond	$\omega^{\upsilon}{}_{\alpha\beta}$	Time evolution is Eq. (4.42). The divergence equations Eq. (4.24). The matrix commutator and curl are determined by Eq. (4.27), since the time dependence of the magnetic field is determined through Eq. (4.50) by the electric field; $\Delta_o \omega_{\upsilon\alpha\beta} = 2 e_\alpha \Delta_o \theta_{\upsilon\beta}$ $\Delta_\upsilon \omega^{\upsilon}{}_{\alpha\beta} = \begin{pmatrix} \omega^{\upsilon}{}_{\alpha\beta} q_\upsilon + \omega_{\upsilon\gamma}{}^{\gamma} \omega^{\upsilon}{}_{\alpha\beta} + \omega_{\alpha\upsilon'}{}^{\upsilon} \omega_{\beta\upsilon}{}^{\upsilon'} - 2\theta_{\upsilon\alpha} \theta^{\upsilon}{}_{\beta} + 2\omega_{\upsilon\alpha} \omega^{\upsilon}{}_{\beta} \\ -\kappa \left(p_{\alpha\beta} - p h_{\alpha\beta} + \frac{\mu - p}{n-1} h_{\alpha\beta} \right) \end{pmatrix}$ $\Delta_\upsilon \omega_{\upsilon'\alpha\beta} - \Delta_{\upsilon'} \omega_{\upsilon\alpha\beta} - \omega_{\upsilon\alpha\gamma} \omega_{\upsilon'\beta}{}^{\gamma} + \omega_{\upsilon\beta\gamma} \omega_{\upsilon'\alpha}{}^{\gamma} - 4\omega_{\upsilon\alpha} \theta_{\upsilon'\beta} + 4\omega_{\upsilon'\alpha} \theta_{\upsilon\beta} = 2 e_\alpha \Delta_o \omega_{\beta\upsilon\upsilon'}$
Tidal magnetic	$\omega_{\upsilon\upsilon'}$	Eq. (4.33) gives the divergence, Eq. (4.46) gives the "curl" and Eq. (4.36) gives the time evolution.

		$\Delta_{\upsilon'}\omega^{\upsilon'}{}_{\upsilon} = -\omega_{\upsilon\upsilon'}\left(\omega^{\upsilon'\alpha}{}_{\alpha} + 2q^{\upsilon'}\right) + 2\theta^{\upsilon'}{}_{\alpha}\omega^{\alpha}{}_{\upsilon\upsilon'} + \tfrac{1}{2}e^{\alpha}e_{\alpha}\Delta_o q_{\upsilon} - e^{\alpha}\Delta_o\theta_{\upsilon\alpha}$ $\begin{pmatrix} +\Delta_{\upsilon}\omega_{\upsilon'\upsilon''} + q_{\upsilon}\omega_{\upsilon'\upsilon''} - 2\omega_{\upsilon\alpha}\omega^{\alpha}{}_{\upsilon'\upsilon''} \\ +\Delta_{\upsilon'}\omega_{\upsilon''\upsilon} + q_{\upsilon'}\omega_{\upsilon''\upsilon} - 2\omega_{\upsilon'\alpha}\omega^{\alpha}{}_{\upsilon''\upsilon} \\ +\Delta_{\upsilon''}\omega_{\upsilon\upsilon'} + q_{\upsilon''}\omega_{\upsilon\upsilon'} - 2\omega_{\upsilon''\alpha}\omega^{\alpha}{}_{\upsilon\upsilon'} \end{pmatrix} = 0$ $\Delta_o\omega_{\upsilon\upsilon'} = \tfrac{1}{2}\left(\Delta_{\upsilon'}q_{\upsilon} - \Delta_{\upsilon}q_{\upsilon'}\right) + 2\theta_{\upsilon}{}^{\alpha}\omega_{\upsilon'\alpha} - 2\omega_{\upsilon\alpha}\theta_{\upsilon'}{}^{\alpha}$

In addition to these strain components, the tidal spin components $\omega_{\upsilon\alpha}$ are not zero in the *player fixed frame model*. Because the gradient of the charge Eq. (3.69) is non-zero whenever these spin components are non-zero, we think of these spin components as the ***tidal charge gradients***. They contribute contributions to Coulomb's law Eq. (4.32) as seen also in Table 4-4. The ***tidal magnetic*** components $\omega_{\upsilon\upsilon'}$ are so named as they contribute to the payoffs Eq. (4.56) in the *normal-form coordinate basis* since by Eq. (3.74):

$$f^{j}{}_{\upsilon\upsilon'} = 2\omega_{\alpha\upsilon\upsilon'}E^{j\alpha} - 2\omega_{\upsilon\upsilon'}E^{jo} \tag{4.57}$$

The dynamics of the tidal magnetic field differs from $\omega_{\alpha\upsilon\upsilon'}$ as seen in Table 4-5. For example we see that the time variation depends on the curl of the acceleration and distinguishes the electric field from the tidal charge gradient. These effects need more detailed study. We may learn more about these dynamics in decision processes than what we currently understand from applications in physics.

What effects do these tidal strains and tidal spins represent in decision processes? We have been able to draw little from game theory as these effects would be absent at equilibrium. A preliminary study by (Thomas & Kane, 2010) suggests that the charge gradient provides the possibility for players to demonstrate interdependent and independent behaviors. We will review that work in the next chapter, updating it to conform to the *decision process theory* outlined here using the *player fixed frame model*.

4.4.4 Distinctions—inertial fields

The *orientation potentials* describe tidal spins and tidal strains, the *orientation flux fields,* which relate directly to the geometry of the behavior. Our *decision process theory* provides not only for strains but for components such as $p_{\upsilon\upsilon'}$ of the ***stress tensor*** that, along with the energy density and energy flow, characterize the energy momentum of inertial fields. The ***orientation flux fields*** are key to our *decision process theory*. The effects of frames that change with the *orientation flux fields* provide the basis for an understanding of how energy *attracts* or *interacts* with energy. Since we take energy to be more fundamental than utility or value, we are in fact hoping to provide a deeper understanding of economic value. We see in Table 4-6 that the *orientation flux field* is directly related to the components $p_{\upsilon\upsilon'}$ of the stress tensor along the active directions. With the physics analogy as a guide, we see that the metric components g^{00} in the time direction will be determined in part by the equations for the transformations $E^{a}{}_{o}$, which are determined by the exactness conditions Eq. (3.60) and found to depend critically on the *orientation flux field*. This mathematical result produces the physical result of (scalar) gravitational attraction that does conform to Newtonian gravity.

Inertial effects occur not only in the acceleration of the flow of energy, but due to the magnitude of the ***energy density*** μ. The energy density Eq. (4.34) is the sum of the contributions from the electromagnetic fields and the other tidal fields. This conservation of energy provides the basis for converting value from one form to another. The average stress p we characterize as the ***pressure***. The field equations yield a particularly interesting result Eq. (4.35) that algebraically relates the diagonal stress components and energy density to the orientation potentials. These diagonal components and the energy density are typically thought to be non-negative so this result imposes strong constraints on the behavior of the electromagnetic and tidal fields.

© 2012-2015 Gerald H Thomas

Of particular interest in applications will be the behavior of the divergence of the acceleration of energy, Eq. (4.23), where we eliminate the divergence of the *tidal bond* Eq. (4.24) and make use of the algebraic relationship for the diagonal pressure components Eq. (4.35), (*Cf.* exercise 37):

$$\Delta_\upsilon q^\upsilon = \kappa\left(\mu + p - \frac{\mu - p}{n-1}\right) + 2\theta_{\upsilon\alpha}\theta^{\upsilon\alpha} - 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \omega_{\upsilon\upsilon'}\omega^{\upsilon\upsilon'} + \omega^{\upsilon\alpha}{}_\alpha q_\upsilon + q_\upsilon q^\upsilon \tag{4.58}$$

In numerical examples we may start at a point of no acceleration, so the last two terms start at zero. We see that the inertial and compression contributions generate a positive contribution, whereas the two vorticity contributions are negative. It is noteworthy that the magnetic contributions have cancelled and don't appear. It is not the payoffs that drive the system to equilibrium. The forces that drive the system towards the equilibrium would be the inertial or compression forces. This is indicated by the divergence.

Divergence $\Delta_\upsilon q^\upsilon$ is a measure of flow out of a small volume centered at the point of interest. Thus the inertial effects lead to an increase of acceleration as we move away from the point of interest. If the actual flow is stationary so that the forces balance, there must be a compensating force that is negative. This compensating force is the gravitational attraction. The vorticity contributions on the other hand cause the acceleration to decrease as we move away from the point of interest. Such contributions provide a centrifugal effect that is opposite to gravity.

Table 4-6: Player fixed frame model—inertial components

Distinction	Variable	Properties
Energy	q^υ	The stress tensor for both components active, Eq. (4.23) provides the gradients of the acceleration of the flow of energy: $$\tfrac{1}{2}\left(\Delta_{\upsilon'}q_\upsilon + \Delta_\upsilon q_{\upsilon'}\right) = \left(\begin{array}{l} q_\upsilon q_{\upsilon'} + \tfrac{1}{2}\Delta_\upsilon \omega^\gamma{}_{\upsilon'\gamma} + \tfrac{1}{2}\Delta_{\upsilon'}\omega^\gamma{}_{\upsilon\gamma} + 2\omega_{\upsilon\upsilon''}\omega_{\upsilon'}{}^{\upsilon''} \\ +2\left(\theta_{\upsilon\alpha} + \omega_{\upsilon\alpha}\right)\left(\theta_{\upsilon'}{}^\alpha + \omega_{\upsilon'}{}^\alpha\right) + 2\omega^\alpha{}_{\upsilon'\upsilon''}\omega_{\alpha\upsilon}{}^{\upsilon''} + \omega^\alpha{}_{\upsilon'\beta}\omega^\beta{}_{\upsilon\alpha} \\ -\kappa\left(p_{\upsilon\upsilon'} - ph_{\upsilon\upsilon'} + \frac{\mu - p}{n-1}h_{\upsilon\upsilon'}\right) \end{array}\right)$$
Pressure ***Energy density***	$p = \tfrac{1}{n}\left(p_\upsilon^\upsilon + p_\alpha^\alpha\right)$ μ	Algebraic relation between pressure p, which is the average stress and the stress tensor components; Eq. (4.35) is an alternate to Eq. (4.34) for the energy density; Eq. (4.51) provides the time evolution: $$\left(\begin{array}{l} \kappa\left(p^\alpha{}_\alpha - ph^\alpha{}_\alpha - \mu + \frac{\mu - p}{n-1}h^\alpha{}_\alpha\right) + \theta_\upsilon{}^\alpha\theta^\upsilon{}_\alpha + 2\theta^\upsilon{}_\alpha\omega_\upsilon{}^\alpha + \omega_{\upsilon\alpha}\omega^{\upsilon\alpha} \\ -\omega^{\upsilon\beta}{}_\beta q_\upsilon + \tfrac{3}{2}\omega^\upsilon{}_{\upsilon'}\omega_\upsilon{}^{\upsilon'} + \tfrac{3}{2}\omega^\alpha{}_{\upsilon\upsilon'}\omega_\alpha{}^{\upsilon\upsilon'} + \tfrac{1}{2}\omega^{\upsilon\alpha}{}_\beta\omega_{\upsilon\alpha}{}^\beta - \tfrac{1}{2}\omega_{\upsilon\alpha}{}^\alpha\omega^{\upsilon\beta}{}_\beta \end{array}\right) = 0$$ $$\Delta_o\mu = -2p^{\upsilon\alpha}\theta_{\upsilon\alpha}$$
Stress tensor	$p_{\upsilon\upsilon'}$ $p_{\upsilon\alpha}$ $p_{\alpha\beta}$	Eq. (4.53) and (4.52) must be satisfied by the sources: $$\Delta_{\upsilon'}p^{\upsilon\upsilon'} = \mu q^\upsilon + p^{\upsilon\upsilon'}q_{\upsilon'} + \omega_{\upsilon'\alpha}{}^\alpha p^{\upsilon\upsilon'} - \omega^\upsilon{}_{\alpha\beta}p^{\alpha\beta} - 2\omega_{\alpha\upsilon'}{}^\upsilon p^{\upsilon'\alpha} - e_\alpha\Delta_o p^{\upsilon\alpha}$$ $$\Delta_\upsilon p^{\upsilon\alpha} = p^{\upsilon\alpha}q_\upsilon - e_\beta\Delta_o p^{\alpha\beta} + \omega_{\upsilon\beta}{}^\alpha p^{\upsilon\beta} + \omega_{\upsilon\beta}{}^\beta p^{\upsilon\alpha}$$ The stress components internal to the players are not constrained.

For applications, a model must be provided for the stress tensor and energy density. We should specify whether these scalar fields change in time, whether they are related *etc*. The simplest choice is one in which all the stress components are determined by the average stress:

$$\begin{aligned} p_{\upsilon\upsilon'} &= ph_{\upsilon\upsilon'} \\ p_{\alpha\beta} &= ph_{\alpha\beta} \\ p_{\upsilon\alpha} &= ph_{\upsilon\alpha} = 0 \end{aligned} \tag{4.59}$$

This stress tensor defines a ***perfect fluid***. However, we find arguments later that this choice is not consistent with the field equations (section 4.5).

We also need a model for the energy density. An assumption used for physical systems is that the energy density is proportional to the pressure:

$$\mu = \alpha p \tag{4.60}$$

We call the proportionality constant the ***structural coupling*** or ***resilience*** α as the player's response to stresses or forces that lead to actions. The more resilient the system, the higher is the value of α and hence the higher is the energy density.

Even such simple assumptions provide insight. We can create separately a model for the inactive space corresponding to the players and a model for the active space corresponding to the strategic choices. In addition, we anticipate a relationship (Thomas G. H., 2006) between the strain observed in the geometry and the stress on the inertial fields. In many physical systems the stress is proportional to the strain. So in a near perfect fluid, we might have the ***conductivity model*** in which

$$p_{\upsilon\alpha} = \sigma\theta_{\upsilon\alpha} \tag{4.61}$$

In a fluid the proportionality constant is the viscosity. In statics it is Young's modulus. In electromagnetic theory, the strain $\theta_{\upsilon\alpha}$ is the electric field and suggests we call $p_{\upsilon\alpha}$ the ***current of player*** α along the direction υ.

Therefore the stress-strain relationship seems entirely like an entirely reasonable assumption: we call the proportionality constant the ***conductivity*** of the medium since the law is a version of Ohm's Law where the electrical resistance is inversely proportional to the conductivity. Ohm's Law is that an applied voltage generates a proportional current. More generally, we keep the idea that $p_{\upsilon\alpha}$ is the *current of player* α even with zero player charge E_{jo}. This is consistent with electromagnetic phenomena in which both the charges and currents specify the coupling of the fields to the sources. In particular a system can be electrically neutral and still display a current: we shall see that this is the case for *decision process theory*. The *player charges* can vanish yet there can be *player currents* that generate payoff fields.

The inertial fields indicate the reluctance or willingness of the players to make choices in a given direction. They suggest to the player whether or not there is an ***impending event*** requiring them to change their behaviors. This is separate from whether the choices made are *right* in for example a *game theory* sense. When players are willing to move in a given direction we have an effect alluded to in section 2.8. When there is great reluctance to change course, we have an often seen behavior to *stay the course*, no matter how disastrous. We see that the inertial field behaviors are not arbitrary however and obey equations that must be considered as part of the field equations. Though we might in isolation be able to describe where a particular behavior might lead, we can't trust that behavior to be part of the overall solution because of the massive coupling of the effects. We suggest that the mechanisms identified are nevertheless real, as are their resultant coupled behaviors.

4.5 *Quasi-stationary hypothesis*

The equations for the *player fixed frame model* are partial differential equations in the *harmonic coordinates*. Since it is a general theorem (Courant & Hilbert, 1962) that such equations with their boundary conditions have well behaved solutions, with sufficient computing power we can solve these equations for interesting decision processes. Unfortunately mathematical existence proofs of solutions are not the same as practical methods for solutions. We have to carefully specify all of the equations and constraints and evaluate whether solutions are not only feasible but practical. We do that in this section.

As we said in the introduction to this chapter, significant insight is obtained in the study of electromagnetic fields by starting with a study of electro-static and magneto-static phenomena, in which all of the fields are ***static***, which is to say ***stationary*** or independent of time. Therefore it makes sense to add to this requirement to our *player fixed frame model*. Unless stated explicitly otherwise, we make that assumption in this book.

Because there are currents and magnetic fields, the approximation is more properly ***quasi-stationary***. These electro-static and magneto-static solutions can be extended to cover features that appear in fully dynamic systems. Wave phenomena are added by extending these stationary solutions to *harmonic* solutions (***phasors***) that can be solved using the same techniques as used for stationary solutions. In this way, more realistic dynamic solutions are built, along with a deeper understanding of the behaviors of systems of this type.

In this section, and indeed for the remainder of the book, we consider such an approach for the *player fixed frame model*. The resultant solutions will be for any number of active or inactive strategies and any number of players in the *player fixed frame model* in which the orientation potentials in the *co-moving frame* are stationary, which we call the ***quasi-stationary hypothesis***. The resultant *harmonics* we term ***electro-gravitational waves***, as the energy can move back and forth from the electromagnetic field to the gravitational field.

We can transform to the ***central holonomic frame*** in which the metric is independent of time. We call time in that frame the ***central time***. We thus see that time in that frame is an isometry. In the *central holonomic frame* we identify ***harmonic*** solutions for the coordinates (***phasors***). For the *fixed frame model*, in the *central holonomic frame* the *active flow* components are zero, exercise 41, Eq. (4.145).

With the *quasi-stationary hypothesis*, the proper time derivative of all the *co-moving orthonormal frame* orientation potentials, acceleration and inertial fields are zero. In other words, along the streamline, none of the potentials change value. We are in a ***stationary orthonormal frame***. Based on exercises 20 and 21 at the end of the chapter, since these scalar fields are independent of proper time, the differential operators for proper time and proper strategy mutually commute when operating on such scalars. We can then treat the differential equations for the orientation potentials using standard techniques for partial differential equations. We go through each table from sections 4.1 and 4.4 to isolate the equations that are relevant. The algebraic equations require special attention, which we do in section 4.5.4, which leads to the assumption that the shear components $\theta_{\upsilon\alpha}$ are zero (which we equate to very large ***conductivity***) and the *tidal bond* tensors $\omega^{\upsilon}{}_{\alpha\beta}$ are diagonal in the inactive indices.

4.5.1 Stationary scalars

In Table 4-1, if we temporarily assume in this section that the active transformations are independent of time, the exactness conditions Eq. (3.60) and Eq. (3.71) reduce to:

$$\begin{aligned}
\Delta_\upsilon E^a{}_o &= \left(q_\upsilon + 2\theta_\upsilon{}^\alpha e_\alpha\right) E^a{}_o \\
\Delta_\upsilon E^{a\upsilon} &= \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) E^a{}_\upsilon \\
\Delta_\upsilon E^a{}_{\upsilon'} - \Delta_{\upsilon'} E^a{}_\upsilon &= 2\left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) E^a{}_o
\end{aligned} \tag{4.62}$$

These equations have solutions. Because we have the divergence and curl of $E^a{}_\upsilon$, we can solve for these fields, exercise 18. Moreover, the divergence of $E^a{}_o$ leads to a solution for $E^a{}_o$ (*Cf.* exercise 18). We deal next with the acceleration fields.

In Table 4-4, based on the *quasi-stationary hypothesis*, we take both the electric and magnetic fields to be constants in proper time. Taking the electric field independent of time we get the divergence of the magnetic field:

$$\Delta_{\upsilon'}\omega_{\alpha\upsilon}{}^{\upsilon'} = \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} - 2\omega^{\upsilon'}{}_\alpha\omega_{\upsilon\upsilon'} + \omega^{\upsilon'}{}_{\alpha\beta}\omega^\beta{}_{\upsilon\upsilon'} + \omega_{\upsilon'\beta}{}^\beta\omega_{\alpha\upsilon}{}^{\upsilon'} - \kappa p_{\alpha\upsilon} \tag{4.63}$$

We impose the constraint that the magnetic field is independent of proper time, analogous to Kirchhoff's Law, from Eq. (4.50):

$$\Delta_\upsilon\theta_{\upsilon'\alpha} - \Delta_{\upsilon'}\theta_{\upsilon\alpha} = -q_{\upsilon'}\theta_{\upsilon\alpha} + q_\upsilon\theta_{\upsilon'\alpha} - \omega_{\upsilon\alpha}{}^\beta\theta_{\upsilon'\beta} + \omega_{\upsilon'\alpha}{}^\beta\theta_{\upsilon\beta} \tag{4.64}$$

This is more complicated that Kirchhoff's law since the curl is not zero. The curl depends on the acceleration as well as the *tidal bond* tensors. Because we assume all fields are independent of proper time, the time dependent Eq. (4.43) is satisfied identically.

The tidal fields simplify as well. From Table 4-5, we see that the components $\omega_{\upsilon\alpha\beta}$ are independent of time because the electric field is independent of time using Eq. (4.42). We obtain a curl condition on the acceleration from requiring the tidal magnetic field to be independent of time from Eq. (4.36):

$$\Delta_{\upsilon'} q_\upsilon - \Delta_\upsilon q_{\upsilon'} = 4\left(\omega_{\upsilon\alpha}\theta_{\upsilon'}{}^{\alpha} - \omega_{\upsilon'\alpha}\theta_{\upsilon}{}^{\alpha}\right) \tag{4.65}$$

This equation along with Eq. (4.23), Table 4-6 determines the differential equations for the acceleration along the active directions.

There is one equation that is not identically satisfied, Eq. (4.27):

$$\begin{aligned}\Delta_\upsilon \omega_{\upsilon'\alpha\beta} - \Delta_{\upsilon'}\omega_{\upsilon\alpha\beta} &= 2\omega_{\upsilon\alpha}\theta_{\upsilon'\beta} - 2\omega_{\upsilon'\alpha}\theta_{\upsilon\beta} + 2\omega_{\upsilon\beta}\theta_{\upsilon'\alpha} - 2\omega_{\upsilon'\beta}\theta_{\upsilon\alpha} \\ \omega_{\upsilon\alpha\gamma}\omega_{\upsilon'\beta}{}^{\gamma} - \omega_{\upsilon\beta\gamma}\omega_{\upsilon'\alpha}{}^{\gamma} &= -2\omega_{\upsilon\alpha}\theta_{\upsilon'\beta} + 2\omega_{\upsilon'\alpha}\theta_{\upsilon\beta} + 2\omega_{\upsilon\beta}\theta_{\upsilon'\alpha} - 2\omega_{\upsilon'\beta}\theta_{\upsilon\alpha}\end{aligned} \tag{4.66}$$

We have written the independent symmetric and antisymmetric parts that must be satisfied separately.

Finally we look at the equations imposed on the inertial forces Table 4-6 and from Eq. (4.51), impose the condition:

$$\theta_{\upsilon\beta} p^{\upsilon\beta} = 0 \tag{4.67}$$

We need more information to impose further constraints, though we note that the last three equations simplify if the electric field $\theta_{\upsilon\alpha}$ is zero. For a fixed current, the electric field is inversely proportional to the ***conductivity***, Eq. (4.61), so this equation implies very large conductivity. The decision system acts like a “meta” in that changes toward equilibrium occur very quickly. In order to better understand this issue as well as the dynamic behaviors, we consider a broader context in which the active transformations are not independent of time.

4.5.2 Streamline solutions

We propose and here define dynamic ***streamline solutions***. Because the scalar fields are independent of proper time, the coefficient fields in the exactness conditions Eq. (3.60) and in the *harmonic gauge condition* Eq. (3.71) are independent of time. We generalize the solutions Eq. (4.62) by superposing solutions in powers τ^n of the ***streamline proper time scalar field***, with coefficients $X_n{}^a(\upsilon)$ that are scalar functions that depend only on the proper-strategy scalar functions y^υ (see exercises 22 and 24):

$$x^a(\upsilon,\tau) = \sum_n X_n{}^a(\upsilon)\tau^n \tag{4.68}$$

This expansion provides time dependent tangent vectors $E^a{}_o$ and $E^a{}_\upsilon$ (exercise 26) that satisfy the commutation rules (exercises 27 and 29) as long as we meet appropriate constraints (exercise 28).

We motivate this approach using the definition of ***integral curves*** that exist for any vector field Z^a. The integral curve is defined by specifying that its tangent through a point is set by the vector field:

$$\frac{dx^a}{ds} = Z^a\left(x(s)\right) \tag{4.69}$$

These coupled first order equations have a unique solution and provide a definition of the coordinate s associated with the vector field. The coordinate s by its definition moves along the streamline and thus provides a path dependent definition of proper time. We refer to this simply as the ***proper time***, with the understanding that the path is understood to be along the streamline.

We apply this concept to the orthonormal set of vectors and conclude that at each point there will be integral curves associated with each orthonormal vector. In particular there will be an integral curve associated with the flow that defines the *proper time* (a streamline):

$$\Delta_o x^a = E^b{}_o \partial_b x^a = \frac{dx^a}{d\tau} = E^a{}_o\left(x(\tau)\right) \tag{4.70}$$

The scalar field τ is the measure defined along the integral curve with $\Delta_o \tau = 1$ (*Cf.* exercises 24-25).

We compute the tangent vectors from the holonomic scalar function $x^a(y,\tau)$ using the result of exercise 25 in terms of the ***characteristic vector potential*** (or simply ***characteristic potential***) a_υ:

$$\begin{aligned} E^a{}_o &= \partial_\tau x^a \\ E^a{}_\upsilon &= \partial_\upsilon x^a + (2a_\upsilon - \lambda_\upsilon \tau)\partial_\tau x^a \end{aligned} \tag{4.71}$$

We use these expressions as the basis for determining the holonomic scalars in terms of the *proper active strategies* and *proper time*. We note that the time component can initially be set as $E^t{}_o = 1$. If we do that, then the second equation at $\tau = 0$ shows that the initial values of $E^t{}_\upsilon$ are determined by the *characteristic potential*[19] a_υ and the variations of the time field $\partial_\upsilon t$:

$$E^t{}_\upsilon\big|_{\tau=0} = \partial_\upsilon t\big|_{\tau=0} + a_\upsilon \tag{4.72}$$

The *characteristic potential* thus provides the initial values of this frame transformation. If the "curl" of the vector field is given (next paragraph), then different frames would be distinguished only by the divergence of the vector field:

$$\partial_\upsilon a^\upsilon + \lambda_\upsilon a^\upsilon = \psi \tag{4.73}$$

The vector field λ_υ is introduced below in Eq. (4.75). Since decision process results are independent of frame, we are free to choose this arbitrary scalar field to be zero, $\psi = 0$:

$$\partial_\upsilon a^\upsilon + \lambda_\upsilon a^\upsilon = 0 \tag{4.74}$$

This provides a covariant definition of the frame.

To show that solutions exist, we turn to the power series expansion Eq. (4.68). In the expansion, Eq. (4.68), we specify the initial values of the coordinate x^a along the surface specified by $\tau = 0$ as well as the initial flows $E^a{}_o$. We argue that the expansion in terms of powers of the *proper time* scalar function τ can be solved in terms of these initial conditions. We require that the tangent vectors $E^a{}_o$ and $E^a{}_\upsilon$ satisfy the commutation rules for the derivatives, Eq. (3.60), which we leave as exercises 27 and 29. We find that the gradient of the scalar field τ along the proper strategy direction is determined (exercises 24, 25, 28, and 32) if we impose, in addition to the gradient condition Eq. (4.74), a condition on the "curl" of the *characteristic potential* a_υ:

$$\begin{aligned} &\Delta_\upsilon \tau = 2a_\upsilon - \lambda_\upsilon \tau \\ &\lambda_\upsilon = q_\upsilon + 2\theta_\upsilon{}^\alpha e_\alpha \equiv \partial_\upsilon q \\ &\partial_\upsilon\left(\mathrm{e}^q a_{\upsilon'}\right) - \partial_{\upsilon'}\left(\mathrm{e}^q a_\upsilon\right) = \mathrm{e}^q\left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) \end{aligned} \tag{4.75}$$

We redefine the term ***characteristic potential*** to be $-2a_\upsilon \mathrm{e}^q = -2a_\upsilon \exp(q)$[20]. Eq. (4.75) is a striking result since it shows that the gradient of *proper time* is not zero but is linear in *proper time* along the streamline. The proportionality constant is given by the acceleration and product of the charge times the electric field (the "force" due to the electric field). The coefficient $-2a_\upsilon \mathrm{e}^q$ acts like the potential for the magnetic and tidal fields and is determined by them. Our choice of "gauge" Eq. (4.74) is $\partial_\upsilon\left(\mathrm{e}^q a^\upsilon\right) = 0$. Finally we see that the curl of λ_υ must vanish. We look for solutions that enforce this condition and are consistent with the field equations.

[19] The potential here is proportional to the analog of the gravitomagnetic potential of general relativity.

[20] As a subtle reminder, the exponential function is written as e^q using the "function" font to distinguish it from the variables e^α that represent the player charges.

4.5.3 Player ownership rule

We show that the conditions on the "curl" of λ_υ are related to the question of how and in what sense, each player claims ownership of strategies. The scalar field λ_υ is a known function of the acceleration, electric field and charge, so its "curl" is determined. To require that the "curl" is zero imposes a condition on these fields. We enforce the condition that the "curl" of λ_υ, Eq. (4.75), is zero by assuming that the electric field (in the co-moving frame) vanishes, a condition that does not violate any of the field equations:

$$\theta_{\upsilon\alpha} = 0 \tag{4.76}$$

This condition becomes part of our ***quasi-stationary hypothesis***. We motivated this assumption earlier by suggesting that this means for finite currents $p_{\upsilon\alpha}$, we have very large ***conductivity***.

This argument is based on a physics analogy and is an example of non-static behaviors of the ***centrally co-moving hypothesis*** (Thomas G. H., 2006) in which time is inactive (exercise 46). A non-zero shear $\theta_{\upsilon\alpha}$ is expected to give rise to a non-zero stress $p_{\upsilon\alpha}$, roughly proportional to the shear (in physics the proportionality would be the viscosity); the proportionality constant is the reciprocal of the conductivity. The rate of change of the energy density is then proportional to $\theta_{\upsilon\alpha}\theta^{\upsilon\alpha}$, which must be zero if the energy density is *stationary*, Eq. (4.67). This implies that each of the shear components is zero. If the shear components (electric field components) are zero, then the coefficient $\lambda_\upsilon = q_\upsilon$ is the acceleration vector and from Eq. (4.65) the "curl" of the acceleration vector is zero. Because the gradients of the acceleration are not zero, Eq. (4.58), we have a non-zero acceleration field. The acceleration effects are a consequence of our frame of reference. We believe this hypothesis is useful because we focus on the ***vorticity*** of the solutions, which we believe to be the significant new feature of ***decision process theory***.

We note that our argument does not logically require that the stress components (*player current*) $p_\alpha{}^\upsilon$ be zero. In fact, to insure that the energy density be *stationary* Eq. (4.51), we require only that the shear components be small (very large *conductivit*y). The corresponding stress components need not vanish. For each (proper) player α, the stress components $p_\alpha{}^\upsilon$ represent their view of the ***player passion*** to exercise strategy υ. The "divergence" of the payoff for this player, Eq. (4.63), shows that the *player passion* is the source of their view of the *player payoff* in the same way that the electric current is the source of the magnetic field in physics:

$$\Delta_{\upsilon'}\omega_{\alpha\upsilon}{}^{\upsilon'} = \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} - 2\omega^{\upsilon'}{}_{\alpha}\omega_{\upsilon\upsilon'} + \omega^{\upsilon'}{}_{\alpha\beta}\omega^{\beta}{}_{\upsilon\upsilon'} + \omega_{\upsilon'\beta}{}^{\beta}\omega_{\alpha\upsilon}{}^{\upsilon'} - \kappa p_{\alpha\upsilon} \tag{4.77}$$

For a single active strategy, every term but the last depends on the payoffs, which must vanish since the payoffs $\omega_{\alpha\upsilon\upsilon'}$ are antisymmetric in the active proper strategies, of which there is only one. This implies for a *single active strategy* that

$$p_{\alpha\upsilon} = 0 \tag{4.78}$$

For two or more active strategies however, it is possible for the *player passion* to be non-zero.

The next question is to determine the appropriate equations of motion for the *player passion*. We tentatively consider the following ***player ownership rule*** for *player passions* (*Cf.* chapter 6): each player is accountable and hence owns only his or her own strategies $\upsilon \in \mathcal{S}_\alpha$, where $\mathcal{S}_\alpha$ is the set of strategies owned by α. By this we mean there can be no *player passion* for a strategy that is not owned by player α, or equivalently $p_\alpha{}^\upsilon = 0$ for every strategy not owned $\upsilon \notin \mathcal{S}_\alpha$. We may have players that own no strategies: we call them ***dependent players***. A player that owns at least one strategy is therefore a ***non-dependent player***. To insure that these distinctions carry a meaning over the space and time of the decision process, a sufficient assumption is that the "curl" of the *player passion* is zero. Then there would be a coordinate surface whose normal aligned with each owned strategy. All such surfaces would be in the subspace the player does not own.

In the next section we show that such equations simplify further because the *tidal bond* $\omega_{\upsilon\alpha\beta}$ can be transformed to a diagonal matrix using a constant transformation. That being the case, we see that the boundary condition for a *dependent player* α, is that $p_\alpha{}^\upsilon = 0$ on an initial surface for every strategy υ and so must propagate to zero everywhere. If a player starts with no ownership of any strategy on an initial surface, that remains true at all other points of space in the special case that the "curl" vanishes.

In a similar way we investigate strategies that a *non-dependent player* owns. When the "curl" vanishes, the integral curves that result from Eq. (4.79) for each strategy will propagate from the initial boundary surface to new surfaces defined by the ***player ownership potential*** $p_\alpha : p_{\alpha\upsilon} = \partial_\upsilon p_\alpha$. These potential surfaces act like "coordinates" of what is possible for the player to influence through her *player passion*. Orthogonal to these surfaces will be the strategies that the player does not own.

Notwithstanding such nice features, we find these assumptions not totally compelling and overly restrictive. The fundamental concept is that the *player passion* is the idiosyncratic view of that player of the stresses all other players feel and this gives rise to the player's idiosyncratic view of their payoff. However, a conclusion that we can draw is the virtue of finding such coordinate surfaces: in other words we need to determine whether the *player passion* in some sense corresponds to a vector field with zero curl.

The equations that determine the *player current* are determined using the "divergence" Eq. (4.52) for *stationary* stresses, along with a tensor $X_{\alpha\upsilon\upsilon'}$ determined by the "curl" from the ***conductivity model.*** That assumption is that the current is proportional to the electric field, Eq. (4.61), and the equation for that field, Eq. (4.50) is given below:

$$\begin{aligned}
&\partial_\upsilon p_\alpha{}^\upsilon = p_\alpha{}^\upsilon q_\upsilon + \omega_{\upsilon\alpha\beta} p^{\beta\upsilon} + \Theta_\upsilon p_\alpha{}^\upsilon \\
&\partial_\upsilon p_{\upsilon'\alpha} - \partial_{\upsilon'} p_{\upsilon\alpha} = q_\upsilon p_{\upsilon'\alpha} - q_{\upsilon'} p_{\upsilon\alpha} - \omega_{\upsilon\alpha}{}^\beta p_{\upsilon'\beta} + \omega_{\upsilon'\alpha}{}^\beta p_{\upsilon\beta} \equiv X_{\alpha\upsilon\upsilon'}
\end{aligned} \tag{4.79}$$

To get small electric field we must go farther and say that the *conductivity* is very large. If the conductivity is not constant there will be an additional term in the second equation.

We see that the "curl" $X_{\alpha\upsilon\upsilon'}$ need not vanish, though it might be instructive to consider a model in which it does. We will denote this as the ***ownership model***, loosely related to the discussion above. There is no assumption that the electric field is proportional to the *player current*: we simply assert that $X_{\alpha\upsilon\upsilon'} = 0$. This model does not make the conductivity assumption.

For the *conductivity model*, there also exists a potential (Exercise 67) that can be used to define coordinate surfaces without assuming that $X_{\alpha\upsilon\upsilon'}$ is identically zero. The potential is found by identifying an integrating factor for the *player passion*.

This leaves us with two quite distinct and useful models for the *player passion*, both defined in terms of potential fields: the *ownership model* and the *conductivity model*. We will study both. We assert that in each case these potentials provide a unique and well defined set of definitions of ***player ownership*** based on the coordinate surfaces defined by their potentials. The strength of ownership is a measure of the ***passion*** the player brings to the decision. This description includes the *conductivity model* where the "curl" does not vanish (Cf. Exercise 67).

We see the connection to ownership as follows. For the pure strategies the player has control over, she is indifferent to which one should be picked. The choice can be made on the basis of what makes the most strategic sense: in game theory this choice is based on an optimization strategy. We think of the surfaces of constant player passion in the same way: they describe choices for which the player is indifferent. The passion vector for each player is normal to her ***indifference surface***. The verbiage makes some sense because the opposite of passion would be indifference.

4.5.4 Inactive stress equations

The tidal ***bond strain*** components reflect the degree to which players cooperate as opposed to decisions that a player influences through their intentionality. The assumption of zero electric field components simplifies the constraint Eq. (4.66) on the tidal stress components:

$$\begin{aligned}\Delta_\upsilon \omega_{\upsilon'\alpha\beta} - \Delta_{\upsilon'}\omega_{\upsilon\alpha\beta} = 0 \\ \omega_{\upsilon\alpha}{}^{\gamma}\omega_{\upsilon'\beta\gamma} = \omega_{\upsilon'\alpha}{}^{\gamma}\omega_{\upsilon\beta\gamma}\end{aligned} \tag{4.80}$$

This says that the *tidal bond* matrices are derivable from a potential and mutually commute. Symmetric matrices that mutually commute can be put into diagonal form with a common transformation. We identify the class of solutions that becomes part of our ***quasi-stationary hypothesis*** in which we impose a diagonal form on the *tidal bond* with a condition on the stresses:

$$\begin{aligned}\kappa p_{\alpha\beta} = \omega_{\alpha\upsilon'}{}^{\upsilon}\omega_{\beta\upsilon}{}^{\upsilon'} + 2\omega_{\upsilon\alpha}\omega^{\upsilon}{}_{\beta} + \eta_\upsilon \omega^{\upsilon}{}_{\alpha\beta} + \kappa\pi_{\alpha\beta} \\ \alpha \neq \beta \Rightarrow \pi_{\alpha\beta} = 0\end{aligned} \tag{4.81}$$

Since the *bond strain* matrices have zero "curl", they can be derived from a potential, $\omega_{\upsilon\alpha\beta} = \Delta_\upsilon \varphi_{\alpha\beta}$. To obtain diagonal compression matrices, the ***strategic viscosity*** η_υ is arbitrary and the ***reduced pressure tensor*** $\pi_{\alpha\beta}$ is diagonal. The divergence condition Eq. (4.24) then sets the diagonal components:

$$\partial_\upsilon \partial^\upsilon \varphi_{\alpha\beta} = \left(q_\upsilon + \partial_\upsilon \varphi_\gamma{}^\gamma - \eta_\upsilon\right)\partial^\upsilon \varphi_{\alpha\beta} - \kappa\left(\pi_{\alpha\beta} - p h_{\alpha\beta} + \frac{\mu - p}{n-1} h_{\alpha\beta}\right) \tag{4.82}$$

For *stationary* tensors, we use exercise 20, Eq. (4.125) and use ordinary derivatives. The resultant *bond strain* matrices are diagonal and are derived from a potential so both equations in Eq. (4.80) are satisfied.

The result can be generalized by transforming the equations using a constant non-singular transformation. If this same transformation is applied to the initial coordinate basis, then we can align each inactive strategy j with a corresponding proper inactive strategy α_j. The equations for the frame transformations Table 4-1 then maintain that choice:

$$\partial_\upsilon E_{j\alpha} = -\omega_{\upsilon\alpha}{}^{\beta} E_{j\beta} \tag{4.83}$$

For the inactive strategy $\alpha = \alpha_j$, the coordinate system exhibits compression or expansion, set by the diagonal components of the *bond* matrix. The measure or value of the choices depends on the position in space.

The components $\alpha \neq \alpha_j$ that start at zero remain at zero. This provides a useful interpretation of the inactive metric Eq. (3.70) for distinct players $j \neq k$:

$$\gamma_{jk} = E_{j\alpha}E_{j\beta}h^{\alpha\beta} + E_{jo}E_{ko} = E_{jo}E_{ko} \tag{4.84}$$

The inactive metric represents the degree of cooperation between the two players and is equal to the overlap of the ***player interest flows*** (player flows or charges). This is after making an appropriate rotation of frames to identify the "diagonal" players.

We achieve additional insight by looking at the ***tidal shear*** equation defined by subtracting from the compression tensor a multiple of the diagonal matrix so that the resultant ***bond shear*** tensor has zero trace (exercise 8, chapter 5):

$$\partial_\upsilon \partial^\upsilon \sigma_{\alpha\beta} = \left(\partial_\upsilon q + \partial_\upsilon \Theta - \eta_\upsilon\right)\partial^\upsilon \sigma_{\alpha\beta} - \kappa\left(\pi_{\alpha\beta} - \tfrac{1}{n_i} h_{\alpha\beta}\pi^{\gamma}{}_{\gamma}\right) \tag{4.85}$$

We see that the *bond compression* $\Theta_\upsilon = \partial_\upsilon \Theta = h^{\alpha\beta}\omega_{\upsilon\alpha\beta}$, *acceleration* $q_\upsilon = \partial_\upsilon q$, *strategic viscosity* η_υ and *reduced pressure* tensor components $\pi_{\alpha\beta}$ determine the shear. The complexities of the other tensor fields show up explicitly in the expression for the compression Eq. (4.34):

$$-\partial_\upsilon \partial^\upsilon \Theta = \kappa\mu - \tfrac{1}{2}\omega_{\alpha\upsilon\upsilon'}\omega^{\alpha\upsilon\upsilon'} - 3\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \tfrac{3}{2}\omega_{\upsilon\upsilon'}\omega^{\upsilon\upsilon'} - \tfrac{1}{2}\partial^\upsilon \sigma^{\alpha\beta}\partial_\upsilon \sigma_{\alpha\beta} - \tfrac{1}{2}\frac{n_i + 1}{n_i}\partial_\upsilon \Theta \partial^\upsilon \Theta \tag{4.86}$$

We have a "gravitational" equation for the ***bond compression potential*** Θ. By its definition, the compression potential, written as $\ln V$, describes the change of a volume element V. As we move in the space of the active strategies, this volume element changes; the more compression, the smaller the volume and conversely. This is clearly a property of the geometry of space in *decision process theory*, not an analogy.

The positive sources for *bond compression* are the energy density, which is determined by the pressure, the magnetic fields (and the electric fields, but we take these components to be zero), the *bond shear* components and the *bond compression* gradients. The negative sources, which provide a type of anti-gravity, are the tidal charge gradient fields and the tidal vorticity fields.

When the energy density $\mu = \alpha p$ is proportional to the average pressure, the size of the average pressure is set by Eq. (4.35):

$$\left(h^{\alpha}{}_{\alpha} + \frac{\left(h^{\upsilon}{}_{\upsilon} - 1\right)\alpha + h^{\alpha}{}_{\alpha}}{n-1} \right) \kappa p = \begin{pmatrix} \kappa \pi^{\alpha}{}_{\alpha} + \eta_{\upsilon}\Theta^{\upsilon} - q_{\upsilon}\Theta^{\upsilon} + \frac{1}{2}\omega^{\upsilon\alpha\beta}\omega_{\upsilon\alpha\beta} - \frac{1}{2}\Theta_{\upsilon}\Theta^{\upsilon} \\ + \frac{1}{2}\omega_{\alpha\upsilon\upsilon'}\omega^{\alpha\upsilon\upsilon'} + 3\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} + \frac{3}{2}\omega^{\upsilon\upsilon'}\omega_{\upsilon\upsilon'} \end{pmatrix} \tag{4.87}$$

If there is at least one active strategy, the coefficient of the average pressure on the left is positive. For the pressure to be positive, the terms on the right must sum up to be positive as well. The average pressure is thus determined by the scalar potentials that are determined by the field equations and functions we can set by model assumptions, the viscosity vector η_{υ} and the reduced pressure components $\pi_{\alpha\beta}$.

Finally, it is worth noting that for a single active strategy, we need not impose the constraint Eq. (4.81) on the stress since the "curl" and commutator relations Eq. (4.80) are satisfied identically. We can choose to make either the strains or the stresses have a simple form.

4.5.5 Active stress equations

We are given a choice for the viscosity vector η_{υ} and the reduced pressure components $\pi_{\alpha\beta}$. We compute the average pressure from Eq. (4.87) and the player current $p_{\alpha}{}^{\upsilon}$ from Eq. (4.79). We show in this section that the field equations determine the remaining stress components, which are the active stresses $p_{\upsilon\upsilon'}$. We use the fact that the acceleration vector q_{υ} has zero "curl" (section 4.5.3) so the vector field is the gradient of a scalar field $q_{\upsilon} = \partial_{\upsilon} q$. Furthermore, the divergence is determined by the scalar fields and the average pressure:

$$\partial^{\upsilon}\partial_{\upsilon} q = q_{\upsilon}q^{\upsilon} + q_{\upsilon}\partial^{\upsilon}\varphi^{\alpha}{}_{\alpha} - \omega_{\upsilon\upsilon'}\omega^{\upsilon\upsilon'} - 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} + \kappa\left(\mu + p - \frac{\mu - p}{n-1} \right) \tag{4.88}$$

We assume solutions satisfy the ***Cauchy-Kowalewsky existence theorem*** (Courant & Hilbert, 1962, p. 39).

In addition to the field equation for the divergence of the acceleration vector, we have a more general set of relations involving the gradients, Eq. (4.23), which demonstrate that the acceleration vectors, along with the other scalar fields determine the active stress components (but see Exercise 66):

$$\kappa p_{\upsilon\upsilon'} = \begin{pmatrix} -\partial_{\upsilon}\partial_{\upsilon'} q + \partial_{\upsilon} q \partial_{\upsilon'} q + \partial_{\upsilon}\varphi_{\alpha}{}^{\beta}\partial_{\upsilon'}\varphi_{\beta}{}^{\alpha} - \partial_{\upsilon}\partial_{\upsilon'}\varphi^{\gamma}{}_{\gamma} \\ +2\omega_{\upsilon\upsilon''}\omega_{\upsilon'}{}^{\upsilon''} + 2\omega_{\upsilon\alpha}\omega_{\upsilon'}{}^{\alpha} + 2\omega_{\alpha\upsilon}{}^{\upsilon''}\omega^{\alpha}{}_{\upsilon'\upsilon''} + \kappa\left(p - \frac{\mu - p}{n-1} \right) h_{\upsilon\upsilon'} \end{pmatrix} \tag{4.89}$$

We have moved significantly away from the concept of a perfect fluid, Eq. (4.59). In any given solution we need to verify that the stresses make sense. The price we pay for having a relatively simple structure for the strains and vorticity components in the *player fixed frame model* is that the energy-momentum stresses deviate from the ideal fluid. The deviations are mandated by the presence of non-zero payoff fields and charge gradient fields, as in the expression Eq. (4.81) for the inactive stress components. These internal fields would not be expected to have a perfect fluid behavior. We conclude from this brief analysis that we have identified the essential equations for the *player fixed frame model*. They form a consistent set with what we expect to be non-trivial solutions.

4.5.6 Frame wave equation

In addition to the *stationary* scalar equations based on the *quasi-stationary hypothesis*, we extract a time dependent wave equation that applies to the *player fixed frame model*. We believe the result justifies the effort. With the assumption of zero electric field, the coefficients of the expansion (4.68) are determined by the *harmonic gauge condition* Eq. (3.71):

$$\left(1+e_\alpha e^\alpha\right)\Delta_o E^{ao}+\Delta_\upsilon E^{a\upsilon}-\left(q^\upsilon+\omega^{\upsilon\alpha}{}_\alpha\right)E^a{}_\upsilon=0 \quad (4.90)$$

We have incorporated the conditions Eq. (3.60) in the properties of the *proper time* scalar field τ. We demonstrate solutions to this equation using a power series expansion. The basic idea is to expand each term in Eq. (4.90) and equate terms with the same power:

$$\begin{aligned}&\left(1+e_\alpha e^\alpha\right)\Delta_o E^{ao}=\left(1+e_\alpha e^\alpha\right)\sum_{n=0}X_{n+2}{}^a\left(n+2\right)\left(n+1\right)\tau^n\\&-\left(q^\upsilon+\omega^{\upsilon\alpha}{}_\alpha\right)E^a{}_\upsilon=-\left(q^\upsilon+\omega^{\upsilon\alpha}{}_\alpha\right)\left(\sum_{n=0}\partial_\upsilon X_n{}^a\tau^n-\sum_{n=0}q_\upsilon X_n{}^a n\tau^n+\sum_{n=0}2a_\upsilon X_{n+1}{}^a\left(n+1\right)\tau^n\right)\end{aligned} \quad (4.91)$$

The divergence (which is the ordinary derivative when acting on *stationary* fields) of the tangent vector $E^a{}_\upsilon$ has several terms:

$$\Delta^\upsilon E^a{}_\upsilon=\begin{pmatrix}\sum_{n=0}\partial^\upsilon\partial_\upsilon X_n{}^a\tau^n-\sum_{n=0}q^\upsilon\partial_\upsilon X_n{}^a n\tau^n+\sum_{n=0}2a^\upsilon\partial_\upsilon X_{n+1}{}^a\left(n+1\right)\tau^n\\-\sum_{n=0}\partial^\upsilon\left(q_\upsilon X_n{}^a\right)n\tau^n+\sum_{n=0}q^\upsilon q_\upsilon X_n{}^a n^2\tau^n-\sum_{n=0}2a^\upsilon q_\upsilon X_{n+1}{}^a\left(n+1\right)^2\tau^n\\+\sum_{n=0}\partial^\upsilon\left(2a_\upsilon X_{n+1}{}^a\right)\left(n+1\right)\tau^n-\sum_{n=0}2q^\upsilon a_\upsilon X_{n+1}{}^a\left(n+1\right)\left(n\right)\tau^n\\+\sum_{n=0}4a^\upsilon a_\upsilon X_{n+2}{}^a\left(n+2\right)\left(n+1\right)\tau^n\end{pmatrix} \quad (4.92)$$

In the sums we changed the summation index in order to identify the common powers.

As an example of the process, we start with arbitrarily picking the first two functions $X_0{}^a\left(\upsilon\right)$ and $X_0{}^a\left(\upsilon\right)$. Based on matching powers of τ we obtain the next term:

$$X_2{}^a=\frac{1}{2\left(1+e_\alpha e^\alpha+4a^\upsilon a_\upsilon\right)}\left(-\partial^\upsilon\partial_\upsilon X_0{}^a+\left(q^\upsilon+\omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_0{}^a+\left(6a_\upsilon q^\upsilon+2a_\upsilon\omega^{\upsilon\alpha}{}_\alpha\right)X_1{}^a-4a^\upsilon\partial_\upsilon X_1{}^a\right) \quad (4.93)$$

We see that the second term in the series is determined by the first two terms: the position and the initial value of $E^a{}_o$. Though more complicated than solving differential equations with constant coefficients, it is clear that the same pattern maintains. The coefficient $X^a{}_{n+2}$ will depend on the functions $X^a{}_n$ and $X^a{}_{n+1}$ and their gradients. The general case is exercise 34. This demonstrates that given arbitrary fields $\left\{X^a{}_0 \quad X^a{}_1\right\}$, all other fields $X^a{}_n$ are determined.

The power series demonstrates that there are solutions to the ***frame wave equation*** Eq. (4.90), based on given initial conditions. We expected this based on general theorems on elliptic partial differential equations (Courant & Hilbert, 1962). To summarize, we write the wave equation as a partial differential equation (exercise 34):

$$\begin{pmatrix}\left(1+e_\alpha e^\alpha+4a^\upsilon a_\upsilon-4a^\upsilon q_\upsilon\tau+q^\upsilon q_\upsilon\tau^2\right)\partial_\tau{}^2x^a+\partial_\upsilon\partial^\upsilon x^a+2\left(2a^\upsilon-q^\upsilon\tau\right)\partial_\upsilon\partial_\tau x^a\\+\left(2\partial_\upsilon a^\upsilon-2\omega^{\upsilon\alpha}{}_\alpha a_\upsilon-4a^\upsilon q_\upsilon+q_\upsilon\omega^{\upsilon\alpha}{}_\alpha\tau-\partial_\upsilon q^\upsilon\tau+2q_\upsilon q^\upsilon\tau\right)\partial_\tau x^a-\left(q^\upsilon+\omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon x^a\end{pmatrix}=0 \quad (4.94)$$

This differs from the usual wave equation because of the time and strategy dependence of the coefficients. These differences make it impossible to find solutions that factor into a function of *proper time* and a function of position.

The coefficients become independent of ***central time*** $\bar{\tau} = \tau \mathrm{e}^{q}$ (exercise 54) when we transform to the ***central holonomic frame*** (exercises 41-47):

$$\begin{pmatrix} \delta^{\upsilon\upsilon'}\bar{\partial}_{\upsilon'}\bar{\partial}_{\upsilon}x^{a} - \left(1+e_{\alpha}e^{\alpha}+4a^{\upsilon}a_{\upsilon}\right)\mathrm{e}^{2q}\,\bar{\partial}_{\bar{\tau}}{}^{2}x^{a} + \left(q^{\upsilon}+\omega^{\upsilon\alpha}{}_{\alpha}\right)\bar{\partial}_{\upsilon}x^{a} \\ -4a^{\upsilon}\,\mathrm{e}^{q}\,\bar{\partial}_{\upsilon}\bar{\partial}_{\bar{\tau}}x^{a} + 2\left(q^{\upsilon}+\omega^{\upsilon\alpha}{}_{\alpha}\right)\mathrm{e}^{q}\,a_{\upsilon}\bar{\partial}_{\bar{\tau}}x^{a} \end{pmatrix} = 0 \tag{4.95}$$

The wave equation has the feature that linear combinations of solutions are again solutions: the equations are linear in x^{a}. In general, solutions to these equations will reflect oscillations as well as attenuation (and growth). Such behaviors can be studied by considering phasor solutions that correspond to a fixed frequency $x^{a} \propto e^{i\varpi\bar{\tau}}$, exercise 55.

The solution to Eq. (4.95) equation provides the transformation between two holonomic coordinate bases: the *normal-form coordinate basis* and the *central holonomic frame coordinate basis* (section 3.6). A phasor solution can be thought of as reflecting a rotating basis. The rotation effects are transmitted from one point in space to another as a wave that is possibly attenuated. Since these effects are present in the coordinate transformation, we should see these effects in other tensor quantities such as the metric and curvature of space-time. In the next section we consider the *harmonic* behaviors implied by each of these equations.

4.5.7 Harmonics

We can construct general solutions from linear combinations of specific solutions. As a step to identify specific solutions that might be of interest, note that any solution will be generated once we specify the boundary conditions on a timelike hypersurface $\tau = 0$. To make the discussion explicit (*Cf.* chapter 8), we imagine that there are three active (proper) spatial directions $\upsilon \in \{x \quad y \quad z\}$, though the results hold for all cases of one or more active strategies. Based on the boundary conditions we then have the active coordinates $x^{a}(x,y,z,\tau)$ at all space time. In particular we have the active coordinates along the space like hypersurface $z = 0$. We can decompose any general function of the remaining three variables $\{x,y,\tau\}$ into a Fourier or *harmonic* series in τ at each transverse point $\{x,y\}$. This might be especially insightful if the behaviors in the space directions $\{x,y\}$ are periodic for example. An analogy from electrical engineering would be a wave guide in which the transverse space direction behaviors are strongly influenced by the geometry of the guide. The behavior along the time direction can be resolved into a continuum of *harmonic* terms starting with a zero frequency (linear) contribution of the form $U + V\tau$, and including sine and cosine terms, $\sin\varpi\tau$ and $\cos\varpi\tau$, respectively. A good deal of understanding of the general solution can be obtained based on expectations of the behavior of the individual *harmonic* contributions.

Conversely, we can start with any *harmonic* contribution along the space like hypersurface $z = 0$, taking specific *harmonics* for the transverse directions such as $\sin mx \sin my$ and multiplying by the time *harmonic* such as $\sin\varpi\tau$. We expect that with suitable boundary conditions, there will be a solution for each of these *harmonics* and that the superposition of such *harmonics* will allow us to reconstruct the general solution.

Though the two approaches should be equivalent, they may differ in their ability to deliver accurate results using numerical approximations. For example, the successive terms in Eq. (4.138) for the power series solutions (exercise 33) involve gradients of previous fields. We expect small errors to grow as we compute higher order terms. We will show below that a consideration of *harmonics* can be based on polynomials with the higher order coefficients set to be small (zero) with the possibility of the lower order terms being computed without the same loss of accuracy (see exercise 35). Based on these considerations, we examine *harmonic* solutions, emphasizing that the *harmonics* are the usual Fourier series on the hypersurface $z=0$, but away from that hypersurface the behaviors follow from the full partial differential equation Eq. (4.94).

The zero frequency *harmonic* solution of Eq. (4.94) is linear in *proper time*:

$$x^a = U_0{}^a + V_0{}^a \tau \tag{4.96}$$

The coefficients are functions of position. Using this as a trial solution, we obtain two coupled partial differential equations for the coefficients:

$$\begin{aligned}&\partial_\upsilon \partial^\upsilon U_0{}^a - \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon U_0{}^a + 4a^\upsilon \partial_\upsilon V_0{}^a + 2\left(\partial_\upsilon a^\upsilon - \omega^{\upsilon\alpha}{}_\alpha a_\upsilon - 2a^\upsilon q_\upsilon\right)V_0{}^a = 0\\ &\partial_\upsilon \partial^\upsilon V_0{}^a - \left(3q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon V_0{}^a + \left(q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + 2q_\upsilon q^\upsilon\right)V_0{}^a = 0\end{aligned} \tag{4.97}$$

This generalizes the approach we started with in Eq. (4.62) at the beginning of the section.

It suggests to us a novel approach. We can have a solution constant in time, linear in time and in general a superposition of ***harmonic polynomials*** of degree N. The *harmonic polynomials* generalize the notion of phasor solutions. In the limit that the degree N goes to infinity, assuming that the limit is sufficiently well behaved, we obtain the ***quasi-stationary harmonics***. As mentioned, because of the complexity of the differential equations, these *harmonic polynomials* will not in general be a simple factor of $e^{i\varpi\tau}$ and a function of position away from the initial space like hypersurface, which we have chosen to illustrate above as $z=0$ for three active strategies. We will however be able to transform to a frame in which we obtain *phasor* solutions defined in the usual way. First however we analyze the situation in our current frame of reference.

Sticking with this illustrative example, a *phasor* solution on $z=0$ would be linear combinations of $\sin\varpi\tau$ and $\cos\varpi\tau$ multiplied by *harmonics* $V_\varpi{}^a(x,y)$ and $U_\varpi{}^a(x,y)$ respectively for the transverse strategies along $\{x,y\}$. We can approximate these *harmonic* functions with *harmonic polynomials* of degree N by taking the real and imaginary parts of the exponential power series expansions of the complex phase $e^{i\varpi\tau}$ and truncating after N terms for $\upsilon_0 = \{x, \quad y, \quad z=0\}$:

$$\begin{aligned}\sin_N(\varpi\tau) &= \operatorname{Im}\sum_{n=0}^{N}\frac{(i\varpi\tau)^n}{n!} \Rightarrow \sin_N(\varpi\tau)_n = \operatorname{Im}\frac{(i\varpi)^n}{n!}\\ \cos_N(\varpi\tau) &= \operatorname{Re}\sum_{n=0}^{N}\frac{(i\varpi\tau)^n}{n!} \Rightarrow \sin_N(\varpi\tau)_n = \operatorname{Re}\frac{(i\varpi)^n}{n!}\end{aligned} \tag{4.98}$$

These power series set the values of the coefficients on the surface υ_0 by requiring $X_{N+1}{}^a = X_{N+2}{}^a = 0$ for polynomials $P_N{}^a(\upsilon,\tau)$, exercise 35. So, on the hypersurface $z=0$, we have:

$$\begin{aligned}X^a{}_n(\upsilon_0) &= V_\varpi{}^a(\upsilon_0)\operatorname{Re}\frac{(i\varpi)^n}{n!} + U_\varpi{}^a(\upsilon_0)\operatorname{Im}\frac{(i\varpi)^n}{n!}\\ Y^a{}_{\upsilon n}(\upsilon_0) &= \partial_\upsilon X^a{}_n(\upsilon_0) = \partial_\upsilon V_\varpi{}^a(\upsilon_0)\operatorname{Re}\frac{(i\varpi)^n}{n!} + \partial_\upsilon U_\varpi{}^a(\upsilon_0)\operatorname{Im}\frac{(i\varpi)^n}{n!}\end{aligned} \tag{4.99}$$

Away from the initial surface, the coefficients will evolve and in general will no longer reflect a pure decomposition into a single *harmonic*. We are led to this complexity because the *harmonic* shapes change with z because the coefficients of the partial differential equations depend on the proper strategies υ. A similar complexity arises in electrical engineering when there are damping effects that depend on z, which arise from reactance contributions.

We convert to first order partial differential equations using $Y^a{}_{\upsilon n}(\upsilon) = \partial_\upsilon X^a{}_n(\upsilon)$. We then have a complete set of coupled ordinary partial differential equations (4.140) that are solved numerically using known techniques (Courant & Hilbert, 1962), using the ***numerical method of lines***, (Wolfram, 1992) in chapter 8. However, this is not always effective for elliptic equations, which characterize our *stationary* equation for two or more active strategies. When there are several inactive strategies, we note that solutions to these coupled partial differential equations may benefit from other lattice techniques such as ***finite element analysis*** (Bhatti, 2005). In any event, the *harmonic polynomials* are extended in a practical way to functions of both the proper strategies and *proper time*, (υ,τ).

Any solution of the partial differential equations can be written as superposition of these extended *harmonic polynomial* solutions. In the limit that the number of polynomial terms goes to infinity, we write any solution as a superposition of *harmonics*.

In electrical engineering, the overall time dependence can be constructed from a superposition of phasor solutions that have defined time dependence $(\sin\varpi\tau,\ \cos\varpi\tau)$ and a computed spatial dependence. These solutions help characterize the behaviors expected. For the wave equation (4.94), we view the *harmonic polynomial* solutions as providing the analogous insight. For any number of active strategies, any superposition of appropriate *harmonic polynomials* will again be a solution. The initial values on the hyperplane will be:

$$x^a(\upsilon_0,\tau) = U_0{}^a(\upsilon_0) + V_0{}^a(\upsilon_0)\tau + \sum_{\varpi}\left(U_{\varpi}{}^a(\upsilon_0)\cos_N\varpi\tau + V_{\varpi}{}^a(\upsilon_0)\sin_N\varpi\tau\right) \tag{4.100}$$

There is a similar expansion for the gradient:

$$\partial_\upsilon x^a(\upsilon_0,\tau) = \partial_\upsilon U_0{}^a(\upsilon_0) + \partial_\upsilon V_0{}^a(\upsilon_0)\tau + \sum_{\varpi}\left(\partial_\upsilon U_{\varpi}{}^a(\upsilon_0)\cos_N\varpi\tau + \partial_\upsilon V_{\varpi}{}^a(\upsilon_0)\sin_N\varpi\tau\right) \tag{4.101}$$

Each *harmonic polynomial* coefficient will evolve to a different spatial dependence, one that is determined from the differential equations. We set the *proper time* behavior on an initial surface in (proper) space and the remaining spatial dependence is determined. An analysis of a complete set of *harmonics* will be equivalent to an analysis of the general solution to the partial differential equations.

The above analysis is in a *co-moving orthonormal frame*. We can transform to the ***central holonomic frame*** in which proper time and proper strategies form the (non-orthonormal) holonomic basis and in which the *fixed frame model* is co-moving (for the active strategies). We then obtain the partial differential equation Eq. (4.95) that can be solved using ***harmonics*** (***phasors***), exercises 55-56, where $x^a = x_{\varpi}{}^a e^{i\varpi\bar{\tau}}$:

$$\begin{pmatrix} \delta^{\upsilon\upsilon'}\bar{\partial}_{\upsilon'}\bar{\partial}_{\upsilon}x_{\varpi}{}^a + \left(\varpi^2 \mathrm{e}^{2q}\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right) + 2i\varpi\,\mathrm{e}^q\, a_\upsilon\left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon\right)\right)x_{\varpi}{}^a \\ +\left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon - 4i\varpi\,\mathrm{e}^q\, a^\upsilon\right)\bar{\partial}_\upsilon x_{\varpi}{}^a \end{pmatrix} = 0 \tag{4.102}$$

We obtain solutions in terms of complex numbers; we superpose solutions with appropriate boundary conditions and take the real part to obtain our final answer. Numerically, either this approach or the previous one gives the same answers. If we have the solution for one such partial differential equation we can use the defining equation for the proper times $\bar{\tau} = \tau\mathrm{e}^q$ to get the other. For example, we see that a steady-state wave in the *central holonomic frame* will have a somewhat different behavior with the proper time:

$$\sin\varpi\bar{\tau} = \sin\left(\varpi\tau\mathrm{e}^q\right) \tag{4.103}$$

The apparent frequency changes depending on the streamline for constant values of the proper time τ because of the variation of the potential q that determines the total acceleration.

Breaking our solution down into *harmonics* provides us necessary and practical insight into the behaviors of decisions in our *decision process theory*. The limitations are the usual sort and are based on the maximum number of terms practical for numerical calculations. If we work in the approximation of a finite number of polynomial terms for Eq. (4.98), or using the *central holonomic frame* Eq. (4.102), we nevertheless expect to be able to ascertain (possibly damped) wave phenomena in our solutions. We expect to observe them more directly in Eq. (4.102). At any point in space, we expect to see *harmonic* behavior. The motion of the peaks (valleys) defines the wave motion, which we expect to propagate at the maximum speed allowed by the theory (corresponding to the speed of light in physical theories). If we create a pulse, then we would expect to see that pulse propagate and possibly dissipate over time and/or space. The number of independent waves will depend on the number of active strategies. In particular, we expect to see wave phenomena even for a single active strategy. In the next section, these general results for any number of strategies will be applied to a single active strategy.

4.6 *Single active strategy dynamic solutions*

In realistic solutions with many active strategies and many players, it may still happen that many of the strategies are effectively not utilized. In this case the solution is an ***equivalent decision process*** with fewer strategies and more players. For each strategy that is not utilized in the sense that its strategy is an isometry, that strategy acts like the inactive strategy of a player, albeit a player that has no active strategies of his own. The simplest case is that in which all but one active strategy is replaced with players who are inactive, but make an impact on the game through their payoffs and other indirect orientation potentials. We call these ***single active strategy*** solutions.

For ***streamline solutions*** with a single active strategy, the partial differential equations have two variables. The number of field equations is reduced since there are no magnetic fields $\omega_{\alpha\upsilon\upsilon'}$ tidal magnetic fields $\omega_{\upsilon\upsilon'}$, or "curl" equations, since these are antisymmetric in the active components. This simplifies the covariant gradients Eq. (4.75):

$$\begin{aligned} &\Delta_o\tau = 1 \\ &a_\upsilon = 0 \\ &\Delta_\upsilon\tau = -q_\upsilon\tau \end{aligned} \tag{4.104}$$

These equations insure that the commutation rules Eq. (3.60) are satisfied. We write the partial differential equations that result from the *harmonic wave equation* Eq. (4.94) for the components of Eq. (4.68), where in the *normal-form coordinate basis*, t is time and u is the active strategy direction[21]:

$$\begin{aligned} &\begin{pmatrix} \left(1+e_\alpha e^\alpha + q^\upsilon q_\upsilon \tau^2\right)\partial_\tau^{\,2} x^a + \partial_\upsilon \partial^\upsilon x^a - 2q^\upsilon \tau\partial_\upsilon \partial_\tau x^a \\ +\left(q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + 2q_\upsilon q^\upsilon\right)\tau\partial_\tau x^a - \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon x^a \end{pmatrix} = 0 \\ &a = t,u \end{aligned} \tag{4.105}$$

The single strategy model is ideal for understanding in better detail how we obtain the *harmonic coordinate* behaviors from these transformations. We expect to gain insight by using the ***harmonic polynomials*** $P_n^a(\upsilon,\tau)$, Eq. (4.98) and the coefficient equations (exercise 35):

$$\begin{aligned} &X_{N+1}{}^a = X_{N+2}{}^a = 0 \\ &\partial_\upsilon\partial^\upsilon X_N{}^a - \left((2N+1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_N{}^a + N\left(q_\upsilon\omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + q^\upsilon q_\upsilon (N+1)\right) X_N{}^a = 0 \\ &\partial_\upsilon\partial^\upsilon X_{N-1}{}^a - \left((2N-1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_{N-1}{}^a + (N-1)\left(q_\upsilon\omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + Nq^\upsilon q_\upsilon\right) X_{N-1}{}^a = 0 \\ &for: s = N-2, N-3, \cdots, 0 \\ &\begin{pmatrix} \partial_\upsilon\partial^\upsilon X_s{}^a - \left((2s+1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_s{}^a + s\left(q_\upsilon\omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + q^\upsilon q_\upsilon (s+1)\right) X_s{}^a \\ +(s+2)(s+1)\left(1+e_\alpha e^\alpha\right) X_{s+2}{}^a \end{pmatrix} = 0 \end{aligned} \tag{4.106}$$

These equations can be successively solved to obtain all of the coefficients once the scalar functions $\{q_\upsilon \quad \omega^{\upsilon\alpha}{}_\alpha \quad e_\alpha \quad \omega_{\upsilon\alpha}\}$ are determined.

From Table 4-1 we get three coupled equations that determine the scalar functions, along with equation for the payoffs:

[21] A similar discussion can be made in the *central holonomic frame* if we use Eq. (4.95). See section 4.8, exercise 60.

$$\begin{aligned}
&\partial_\upsilon E_{j\alpha} = -\omega_{\upsilon\alpha}{}^{\beta} E_{j\beta} \\
&\partial_\upsilon E_{jo} = -q_\upsilon E_{jo} - 2\omega_\upsilon{}^{\alpha} E_{j\alpha} \\
&\partial_\upsilon e_\alpha = -q_\upsilon e_\alpha + 2\omega_{\upsilon\alpha} + \omega_{\upsilon\alpha\beta} e^\beta \\
&f^j{}_{o\alpha} = f^j{}_{\alpha\beta} = 0, \quad f^j{}_{\upsilon\upsilon} = 0 \\
&f^j{}_{o\upsilon} = 2\left(q_\upsilon E^{jo} + \omega_{\upsilon\alpha}\left(E^{j\alpha} + e^\alpha E^{jo}\right) + \omega_{\upsilon\alpha\beta} e^\alpha E^{j\beta}\right)\left(1 - E^k{}_o E_k{}^o\right)
\end{aligned} \tag{4.107}$$

Given the active transformations determined from x^a and the equations above for the inactive vectors, the metric components are determined, Eq. (3.70) and (3.72).

The remaining scalars are given in section 4.4. The tidal charge gradients $\omega_{\upsilon\alpha}$ reflect the ***vorticity*** attributes of the solution and are determined from Table 4-4:

$$\partial_\upsilon \varpi^\upsilon{}_\alpha = 2\omega_{\upsilon\alpha} q^\upsilon - \varpi^\upsilon{}_{\alpha\beta}\omega_\upsilon{}^\beta + \varpi^{\upsilon\beta}{}_\beta \omega_{\upsilon\alpha} \tag{4.108}$$

There are no magnetic or tidal magnetic fields. Because there is only a single active strategy, the *player current* $p_{\alpha\upsilon} = 0$ is zero, section 4.5.3, Eq. (4.78). The tidal forces Table 4-5 determine the divergence of $\varpi^\upsilon{}_{\alpha\beta}$:

$$\partial_\upsilon \varpi^\upsilon{}_{\alpha\beta} = \varpi^\upsilon{}_{\alpha\beta} q_\upsilon + \omega_{\upsilon\gamma}{}^\gamma \varpi^\upsilon{}_{\alpha\beta} + 2\omega_{\upsilon\alpha}\varpi^\upsilon{}_\beta - \kappa\left(p_{\alpha\beta} - p h_{\alpha\beta} + \frac{\mu - p}{n-1} h_{\alpha\beta}\right) \tag{4.109}$$

The inertial forces (using Eq. (4.141) from exercise 36) determine the active diagonal stress component:

$$\kappa p^\upsilon{}_\upsilon = -\varpi^{\upsilon\beta}{}_\beta q_\upsilon + \omega_{\upsilon\alpha}\varpi^{\upsilon\alpha} + \tfrac{1}{2}\varpi^{\upsilon\alpha}{}_\beta \omega_{\upsilon\alpha}{}^\beta - \tfrac{1}{2}\omega_{\upsilon\alpha}{}^\alpha \varpi^{\upsilon\beta}{}_\beta \tag{4.110}$$

We are free to pick the energy density scalar μ and inactive stress scalars $p_{\alpha\beta}$. The average pressure is determined from the diagonal stresses, Eq. (2.41). The inertial forces also determine the divergence of the acceleration, Eq. (4.58):

$$\partial_\upsilon q^\upsilon = q_\upsilon q^\upsilon + \varpi^{\upsilon\beta}{}_\beta q_\upsilon - 2\omega_{\upsilon\alpha}\varpi^{\upsilon\alpha} + \kappa\left(\mu + p - \frac{\mu - p}{n-1}\right) \tag{4.111}$$

We use this to eliminate the divergence in Eq. (4.105):

$$\begin{pmatrix} \left(1 + e_\alpha e^\alpha + q^\upsilon q_\upsilon \tau^2\right)\partial_\tau{}^2 x^a + \partial_\upsilon \partial^\upsilon x^a - 2q^\upsilon \varpi \partial_\upsilon \partial_\tau x^a \\ +\left(q_\upsilon q^\upsilon + 2\omega_{\upsilon\alpha}\varpi^{\upsilon\alpha} - \kappa\left(\mu + p - \dfrac{\mu - p}{n-1}\right)\right)\varpi\partial_\tau x^a - \left(q^\upsilon + \varpi^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon x^a \end{pmatrix} = 0 \tag{4.112}$$

We thus have sufficient equations to determine all unknown scalar fields as well as the *harmonic polynomials* for any index n (see exercise 40).

Our numerical applications will explore these single strategy streamline solutions. We will be able to see how the equations provide insight into decision processes. The streamline solutions of the *player fixed frame model* provide the first step towards a quantitative understanding of dynamic behaviors in our *decision process theory*. In the next chapter we return to the prisoner's dilemma, an important example introduced in section 1.4. We provide a more in-depth analysis in chapter 8 and demonstrate the relationship between individual *player payoffs* and the collective ***vorticity behaviors*** in ***decision process theory***.

4.7 *Outcomes*

In this chapter, we accomplished our goal of deconstructing the theory into its component distinctions. We demonstrated that there is a class of models that corresponds to the stationary behavior of AC circuits in electrical engineering: these are models in which time is inactive and mutually commuting with the player inactive strategies. The resultant equations are partial differential equations that are amenable to numerical solution for any number of players and any number of strategies.

We demonstrated this in detail for the *player fixed frame model*. For this model, the student will have learned that the compact mathematical expression Eq. (3.1) for *decision process theory* expands into a large number of separate and testable components summarized in Table 4-1, Table 4-4, Table 4-5 and Table 4-6. By going into this detail, the student learns the essence of the theory. Based on these detailed expressions, the student will understand how to test the many aspects of the self-consistent framework against observed behaviors in decision processes. The results come together in the *player fixed frame model* with the *quasi-stationary hypothesis*. In the next chapter, we apply this model to the prisoner's dilemma, a model that has sparked interest in the game theory literature.

The attainment of the outcomes of this chapter is facilitated by doing the exercises in the following section. Based on this investment, the student should achieve the more detailed outcomes below.

- One of the goals is to obtain equations for the transformations from the *normal-form coordinate basis* to the *orthonormal coordinate basis*. In section 4.1, the student should understand the time evolution and gradient equations for these transformations and their basis. The inactive Killing vectors $\left\{E_j^{\ o} \quad E_j^{\ \alpha} \quad E_j^{\ \upsilon}\right\}$ are *stationary* for the *player fixed frame model*. By contrast, the active strategy transformations $\left\{E^a{}_o \quad E^a{}_\alpha = e_\alpha E^a{}_o \quad E^a{}_\upsilon\right\}$ depend on the *proper time* through a wave equation. The proper charge e_α however is *stationary*. Sufficient equations exist for the gradients of each *stationary* field so that these fields are determined given initial conditions. The results are summarized in Table 4-1.
- In section 4.2, the field equations are expanded in the co-moving orthonormal coordinate basis. The student will see the resultant equations group into evolution equations, constraint (Codacci) equations and conservation laws, characterized in Table 4-2 and Table 4-3. The Codacci constraints involve only partial derivatives that lie in a surface transverse to the flow of *proper time*.
- In section 4.3, the field equations are expanded in more detail in the *player fixed frame model*, producing the results listed in Table 4-4, Table 4-5 and Table 4-6. The student will see the origin of Maxwell like equations as well as new equations reflecting the tidal forces of the model.
- The student will learn new distinctions based on the field equations of *decision process theory* from section 4.4. In particular the student will be able to identify the persistency associated with players, electromagnetic like fields associated with payoff fields, inertial fields associated with *orientation flux fields* and new effects associated with such tidal fields.
- The various distinctions and their related field equations are brought together in section 4.5 under the *quasi-stationary hypothesis* to provide a basis for *harmonic* solutions along streamlines. These solutions are electro-gravitational waves of *decision process theory*. On the transverse surface, the resultant equations are partial differential equations that are mathematically similar to those of electromagnetic theory. The student should gain an appreciation of how these equations might be solved.
- In section 4.5.7, these equations are coupled partial differential equations, which can be solved numerically using the method of lines and work for any number of active strategies, inactive strategies or players. The equations may be solvable using lattice techniques, though that has not yet been investigated.
- In section 4.6, these equations are specialized to a single active strategy where the equations that need to be solved are coupled ordinary differential equations that fall easily into the category of equations that can be solved using current computing techniques.

4.8 *Exercises*

1. Show that the Fermi derivative as defined in Eq. (4.2) leaves the co-moving orthonormal metric (which is Minkowski) unchanged.

2. Provide the frame equations in the *player fixed frame model* starting from Eq. (3.62).
3. Derive the time dependence of the charge, Eq. (3.59).
4. Derive the space dependence of the charge gradient, Eq. (3.69).
5. Demonstrate that the differential of the mixed 1-form potential is:

$$d\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} = \begin{pmatrix} \left(\Delta_o\left(\theta^\alpha{}_\beta + \omega^\alpha{}_\beta\right) + q^\alpha q_\beta - \Delta_\beta q^\alpha + \left(\theta^\alpha{}_\delta + \omega^\alpha{}_\delta\right)\left(\theta^\delta{}_\beta + \omega^\delta{}_\beta + b^\delta{}_\beta\right)\right)\mathbf{V} \wedge \mathbf{E}^{\boldsymbol{\beta}} \\ \left(-\Delta_\gamma\left(\theta^\alpha{}_\beta + \omega^\alpha{}_\beta\right) + \omega^\delta{}_{\beta\gamma}\left(\theta^\alpha{}_\delta + \omega^\alpha{}_\delta\right) - q^\alpha\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right)\right)\mathbf{E}^{\boldsymbol{\beta}} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \end{pmatrix} \tag{4.113}$$

$\alpha, \beta, \gamma, \cdots \neq o$

6. Use Eq. (4.113) to show that the mixed curvature 2-form is:

$$\mathbf{R}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} = \begin{pmatrix} \begin{pmatrix} \Delta_o\left(\theta^\alpha{}_\beta + \omega^\alpha{}_\beta\right) + q^\alpha q_\beta - \Delta_\beta q^\alpha - q^\gamma \omega^\alpha{}_{\gamma\beta} \\ +\left(\theta^\alpha{}_\delta + \omega^\alpha{}_\delta\right)\left(\theta^\delta{}_\beta + \omega^\delta{}_\beta + b^\delta{}_\beta\right) - b^\alpha{}_\delta\left(\theta^\delta{}_\beta + \omega^\delta{}_\beta\right) \end{pmatrix}\mathbf{V} \wedge \mathbf{E}^{\boldsymbol{\beta}} \\ \begin{pmatrix} -\Delta_\gamma\left(\theta^\alpha{}_\beta + \omega^\alpha{}_\beta\right) + \omega^\delta{}_{\beta\gamma}\left(\theta^\alpha{}_\delta + \omega^\alpha{}_\delta\right) \\ +\omega^\alpha{}_{\delta\beta}\left(\theta^\delta{}_\gamma + \omega^\delta{}_\gamma\right) - q^\alpha\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) \end{pmatrix}\mathbf{E}^{\boldsymbol{\beta}} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \end{pmatrix} \tag{4.114}$$

$\alpha, \beta, \gamma, \cdots \neq o$

7. Extract the following curvature tensor components from Eq. (4.114):

$$R_{\alpha o\beta\gamma} = \begin{pmatrix} \Delta_\beta\left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right) - \left(\theta_{\alpha\delta} + \omega_{\alpha\delta}\right)\omega^\delta{}_{\gamma\beta} - \omega_{\alpha\delta\gamma}\left(\theta^\delta{}_\beta + \omega^\delta{}_\beta\right) - q_\alpha \omega_{\beta\gamma} \\ -\Delta_\gamma\left(\theta_{\alpha\beta} + \omega_{\alpha\beta}\right) + \left(\theta_{\alpha\delta} + \omega_{\alpha\delta}\right)\omega^\delta{}_{\beta\gamma} + \omega_{\alpha\delta\beta}\left(\theta^\delta{}_\gamma + \omega^\delta{}_\gamma\right) - q_\alpha \omega_{\beta\gamma} \end{pmatrix}$$

$$R_{\alpha oo\beta} = \begin{Bmatrix} \Delta_o\left(\theta_{\alpha\beta} + \omega_{\alpha\beta}\right) + q_\alpha q_\beta - \Delta_\beta q_\alpha + \left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right)\left(\theta^\gamma{}_\beta + \omega^\gamma{}_\beta + \phi^\gamma{}_\beta\right) \\ -\omega_{\alpha\gamma\beta} q^\gamma - \phi_{\alpha\delta}\left(\theta^\delta{}_\beta + \omega^\delta{}_\beta\right) \end{Bmatrix} \tag{4.115}$$

$\alpha, \beta, \gamma, \cdots \neq o$

8. Use Eq. (4.7) to show that the transverse components of the 1-form potential have the following differential 2-form:

$$d\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = \begin{pmatrix} \left(\Delta_o \omega^\alpha{}_{\beta\gamma} + \omega^\alpha{}_{\beta\varepsilon}\left(\theta^\varepsilon{}_\gamma + \omega^\varepsilon{}_\gamma\right) + \Delta_\gamma \phi^\alpha{}_\beta - q_\gamma \phi^\alpha{}_\beta + \omega^\alpha{}_{\beta\varepsilon}\phi^\varepsilon{}_\gamma\right)\mathbf{V} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \\ +\left(\Delta_\delta \omega^\alpha{}_{\beta\gamma} + \omega^\alpha{}_{\beta\varepsilon}\omega^\varepsilon{}_{\delta\gamma} + \phi^\alpha{}_\beta\left(\theta_{\delta\gamma} + \omega_{\delta\gamma}\right)\right)\mathbf{E}^{\delta} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \end{pmatrix} \tag{4.116}$$

$\alpha, \beta, \gamma, \cdots \neq o$

9. Use Eq. (4.116) to show that the transverse curvature 2-form is:

$$\mathbf{R}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = d\boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\mathbf{o}} \wedge \boldsymbol{\omega}^{\mathbf{o}}{}_{\boldsymbol{\beta}} + \boldsymbol{\omega}^{\boldsymbol{\alpha}}{}_{\varepsilon} \wedge \boldsymbol{\omega}^{\varepsilon}{}_{\boldsymbol{\beta}}$$

$$\mathbf{R}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = \begin{pmatrix} \begin{pmatrix} \Delta_o \omega^\alpha{}_{\beta\gamma} + \omega^\alpha{}_{\beta\varepsilon}\left(\theta^\varepsilon{}_\gamma + \omega^\varepsilon{}_\gamma\right) - \left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right)q^\alpha + q_\beta\left(\theta^\alpha{}_\gamma + \omega^\alpha{}_\gamma\right) \\ +\Delta_\gamma \phi^\alpha{}_\beta - q_\gamma \phi^\alpha{}_\beta + \omega^\alpha{}_{\beta\varepsilon}\phi^\varepsilon{}_\gamma + \left(\omega^\alpha{}_{\varepsilon\gamma}\phi^\varepsilon{}_\beta - \phi^\alpha{}_\varepsilon \omega^\varepsilon{}_{\beta\gamma}\right) \end{pmatrix}\mathbf{V} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \\ +\begin{pmatrix} \Delta_\delta \omega^\alpha{}_{\beta\gamma} + \omega^\alpha{}_{\beta\varepsilon}\omega^\varepsilon{}_{\delta\gamma} - \left(\theta^\alpha{}_\delta + \omega^\alpha{}_\delta\right)\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) \\ +\omega^\alpha{}_{\varepsilon\delta}\omega^\varepsilon{}_{\beta\gamma} + \phi^\alpha{}_\beta\left(\theta_{\delta\gamma} + \omega_{\delta\gamma}\right) \end{pmatrix}\mathbf{E}^{\delta} \wedge \mathbf{E}^{\boldsymbol{\gamma}} \end{pmatrix} \tag{4.117}$$

$\alpha, \beta, \gamma, \cdots \neq o$

10. Extract the following curvature tensor components from Eq. (4.117):

$$R_{\alpha\beta o\gamma} = \begin{pmatrix} \Delta_o \omega_{\alpha\beta\gamma} + \omega_{\alpha\beta\varepsilon}\left(\theta^{\varepsilon}{}_{\gamma} + \omega^{\varepsilon}{}_{\gamma}\right) + q_{\beta}\left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right) - q_{\alpha}\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) \\ +\Delta_{\gamma}\phi_{\alpha\beta} - q_{\gamma}\phi_{\alpha\beta} + \omega_{\alpha\beta\varepsilon}\phi^{\varepsilon}{}_{\gamma} + \left(\omega^{\alpha}{}_{\varepsilon\gamma}\phi^{\varepsilon}{}_{\beta} - \phi^{\alpha}{}_{\varepsilon}\omega^{\varepsilon}{}_{\beta\gamma}\right) \end{pmatrix}$$

$$R_{\alpha\beta\delta\gamma} = \begin{pmatrix} \Delta_{\delta}\omega_{\alpha\beta\gamma} + \omega_{\alpha\beta\varepsilon}\omega^{\varepsilon}{}_{\delta\gamma} - \left(\theta_{\alpha\delta} + \omega_{\alpha\delta}\right)\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) - \omega_{\varepsilon\alpha\delta}\omega^{\varepsilon}{}_{\beta\gamma} + \phi_{\alpha\beta}\omega_{\delta\gamma} \\ -\Delta_{\gamma}\omega_{\alpha\beta\delta} - \omega_{\alpha\beta\varepsilon}\omega^{\varepsilon}{}_{\gamma\delta} + \left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right)\left(\theta_{\beta\delta} + \omega_{\beta\delta}\right) + \omega_{\varepsilon\alpha\gamma}\omega^{\varepsilon}{}_{\beta\delta} + \phi_{\alpha\beta}\omega_{\delta\gamma} \end{pmatrix} \quad (4.118)$$

$$\alpha, \beta, \gamma, \cdots \neq o$$

11. Use Eq. (4.118), (4.115) and the symmetry relations $R_{\alpha\beta o\gamma} = R_{\gamma o\beta\alpha}$ to establish the time evolution of the transverse orientation potentials:

$$\Delta_o \omega_{\alpha\beta\gamma} = \begin{pmatrix} -\left(\theta_{\gamma\beta} + \omega_{\gamma\beta}\right)_{\backslash\alpha} + \left(\theta_{\alpha\gamma} - \omega_{\alpha\gamma}\right)_{\backslash\beta} - \omega_{\alpha\varepsilon\gamma}\phi^{\varepsilon}{}_{\beta} + \phi_{\alpha\varepsilon}\omega^{\varepsilon}{}_{\beta\gamma} - \omega_{\alpha\beta\varepsilon}\phi^{\varepsilon}{}_{\gamma} \\ -\omega_{\alpha\beta\varepsilon}\left(\theta^{\varepsilon}{}_{\gamma} + \omega^{\varepsilon}{}_{\gamma}\right) \\ -q_{\beta}\left(\theta_{\alpha\gamma} + \omega_{\alpha\gamma}\right) + q_{\alpha}\left(\theta_{\beta\gamma} + \omega_{\beta\gamma}\right) + 2q_{\gamma}\omega_{\alpha\beta} - \Delta_{\gamma}\phi_{\alpha\beta} + q_{\gamma}\phi_{\alpha\beta} \end{pmatrix} \quad (4.119)$$

$$\alpha, \beta, \gamma, \cdots \neq o$$

12. Use the symmetry relation $R_{\alpha\beta\delta\gamma} = R_{\delta\gamma\alpha\beta}$ and Eq. (4.118) to obtain the constraints that are spatial partial differential equations for the orientation potentials:

$$\begin{pmatrix} \Delta_{\delta}\omega_{\alpha\beta\gamma} - \Delta_{\gamma}\omega_{\alpha\beta\delta} - \Delta_{\alpha}\omega_{\delta\gamma\beta} + \Delta_{\beta}\omega_{\delta\gamma\alpha} \\ -2\omega_{\alpha\delta}\theta_{\beta\gamma} - 2\theta_{\alpha\delta}\omega_{\beta\gamma} + 2\omega_{\alpha\gamma}\theta_{\beta\delta} + 2\theta_{\alpha\gamma}\omega_{\beta\delta} + 2\phi_{\alpha\beta}\omega_{\delta\gamma} - 2\phi_{\delta\gamma}\omega_{\alpha\beta} \\ +\omega_{\alpha\beta\varepsilon}\omega^{\varepsilon}{}_{\delta\gamma} + \omega_{\delta\gamma\varepsilon}\omega^{\varepsilon}{}_{\beta\alpha} - \omega_{\alpha\beta\varepsilon}\omega^{\varepsilon}{}_{\gamma\delta} - \omega^{\varepsilon}{}_{\alpha\beta}\omega_{\delta\gamma\varepsilon} \\ -\omega_{\varepsilon\alpha\delta}\omega^{\varepsilon}{}_{\beta\gamma} + \omega_{\varepsilon\alpha\gamma}\omega^{\varepsilon}{}_{\beta\delta} + \omega_{\varepsilon\delta\alpha}\omega^{\varepsilon}{}_{\gamma\beta} - \omega_{\varepsilon\delta\beta}\omega^{\varepsilon}{}_{\gamma\alpha} \end{pmatrix} = 0 \quad (4.120)$$

$$\alpha, \beta, \gamma, \cdots \neq o$$

13. Show that multiplying Eq. (4.120) by $m^{\delta\beta}$ gives:

$$\begin{pmatrix} \Delta_{\delta}\omega^{\delta}{}_{\beta\alpha} - \Delta_{\delta}\omega^{\delta}{}_{\alpha\beta} + \Delta_{\beta}\omega^{\delta}{}_{\alpha\delta} - \Delta_{\alpha}\omega^{\delta}{}_{\beta\delta} - 2\omega_{\alpha}{}^{\delta}\theta_{\delta\beta} - 2\theta_{\alpha}{}^{\delta}\omega_{\delta\beta} + 2\omega_{\alpha\beta}\theta \\ -\omega^{\delta}{}_{\varepsilon\delta}\left(\omega^{\varepsilon}{}_{\alpha\beta} - \omega^{\varepsilon}{}_{\beta\alpha}\right) + 2\phi_{\alpha\delta}\omega^{\delta}{}_{\beta} - 2\omega_{\alpha}{}^{\delta}\phi_{\delta\beta} \end{pmatrix} = 0 \quad (4.121)$$

$$\alpha, \beta, \gamma, \cdots \neq o$$

14. From the form of the energy momentum tensor Eq. (2.74) demonstrate that:

$$h^{\alpha\beta}R_{\alpha\beta} - R_{oo} = 2\kappa\mu$$
$$\alpha, \beta \neq o \quad (4.122)$$

15. Use Eq. (4.122) and the contracted forms for the curvature tensor to show:

$$\Delta_{\alpha}\omega^{\gamma\alpha}{}_{\gamma} + \tfrac{1}{2}\theta^{2} - \tfrac{1}{2}\theta^{\alpha}{}_{\beta}\theta^{\beta}{}_{\alpha} - \tfrac{1}{2}\omega^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} + \tfrac{1}{2}\omega^{\alpha\beta}{}_{\gamma}\omega^{\gamma}{}_{\beta\alpha} - \tfrac{1}{2}\omega^{\alpha}{}_{\beta\alpha}\omega^{\beta\gamma}{}_{\gamma} + \phi^{\alpha}{}_{\beta}\omega^{\beta}{}_{\alpha} = \kappa\mu \quad (4.123)$$
$$\alpha, \beta, \gamma, \cdots \neq o$$

16. Use Eq. (4.123) to show that Eq. (4.13) and the "trace" of Eq. (4.17) are consistent.
17. Derive Eq. (4.21) from Eq. (4.20).
18. Prove that if the "curl" and "divergence" are known, then the fields are determined. Use the ***Cauchy-Kowalewsky existence theorem*** (Courant & Hilbert, 1962, p. 39).
19. Check to see whether the conditions Eq. (2.71) in the co-moving orthonormal coordinate basis provide any new restrictions. Since we have already checked each of the symmetry conditions Eq. (2.72), use the fact that we only need to check the cyclic tensor Eq. (2.73) and derive the forms below to show that no new constraints result:

$$
\alpha,\beta,\gamma,\delta,\varepsilon \neq o
$$

$$
X_{\alpha\beta\gamma\delta} = \begin{pmatrix} \Delta_\beta\left(\omega_{\alpha\delta\gamma}-\omega_{\alpha\gamma\delta}\right)+\Delta_\gamma\left(\omega_{\alpha\beta\delta}-\omega_{\alpha\delta\beta}\right)+\Delta_\delta\left(\omega_{\alpha\gamma\beta}-\omega_{\alpha\beta\gamma}\right) \\ +\left(\omega_{\alpha\beta\varepsilon}+\omega_{\varepsilon\alpha\beta}\right)\left(\omega^\varepsilon{}_{\gamma\delta}-\omega^\varepsilon{}_{\delta\gamma}\right)+\left(\omega_{\alpha\gamma\varepsilon}+\omega_{\varepsilon\alpha\gamma}\right)\left(\omega^\varepsilon{}_{\delta\beta}-\omega^\varepsilon{}_{\beta\delta}\right) \\ +\left(\omega_{\alpha\delta\varepsilon}+\omega_{\varepsilon\alpha\delta}\right)\left(\omega^\varepsilon{}_{\beta\gamma}-\omega^\varepsilon{}_{\gamma\beta}\right) \\ +2\left(\theta_{\alpha\beta}+\omega_{\alpha\beta}+\phi_{\alpha\beta}\right)\omega_{\gamma\delta}+2\left(\theta_{\alpha\gamma}+\omega_{\alpha\gamma}+\phi_{\alpha\gamma}\right)\omega_{\delta\beta}+2\left(\theta_{\alpha\delta}+\omega_{\alpha\delta}+\phi_{\alpha\delta}\right)\omega_{\beta\gamma} \end{pmatrix} \quad (4.124)
$$

$$
X_{o\alpha\beta\gamma} = 2\begin{pmatrix} q_\alpha\omega_{\beta\gamma}+q_\beta\omega_{\gamma\alpha}+q_\gamma\omega_{\alpha\beta}+\Delta_\alpha\omega_{\beta\gamma}+\Delta_\beta\omega_{\gamma\alpha}+\Delta_\gamma\omega_{\alpha\beta} \\ +\omega_{\alpha\delta}\left(\omega^\delta{}_{\gamma\beta}-\omega^\delta{}_{\beta\gamma}\right)+\omega_{\beta\delta}\left(\omega^\delta{}_{\alpha\gamma}-\omega^\delta{}_{\gamma\alpha}\right)+\omega_{\gamma\delta}\left(\omega^\delta{}_{\beta\alpha}-\omega^\delta{}_{\alpha\beta}\right) \end{pmatrix}
$$

20. For any scalar field φ, prove that in the co-moving basis, $\varphi_{|\upsilon\upsilon'}=\varphi_{|\upsilon'\upsilon}$ and hence for the *player fixed frame model*:

$$
\Delta_\upsilon\Delta_{\upsilon'}\varphi-\Delta_{\upsilon'}\Delta_\upsilon\varphi=2\left(\omega_{\upsilon\upsilon'}-e_\alpha\omega^\alpha{}_{\upsilon\upsilon'}\right)\Delta_o\varphi \quad (4.125)
$$

21. For any scalar field φ, prove that in the co-moving basis, $\varphi_{|\upsilon o}=\varphi_{|o\upsilon}$ and hence for the *player fixed frame model*:

$$
\Delta_\upsilon\Delta_o\varphi-\Delta_o\Delta_\upsilon\varphi=\left(q_\upsilon+2\theta_\upsilon{}^\alpha e_\alpha\right)\Delta_o\varphi \quad (4.126)
$$

22. Show that for any scalar function $\varphi(\upsilon)$ that is a function of the proper-active strategies y^υ only, that such a function is independent of *proper time*. In particular demonstrate for each coordinate that

$$
\Delta_o y^\upsilon = 0 \quad (4.127)
$$

23. Show that the gradients of proper-active strategies with respect to proper-active strategies are orthonormal:

$$
\Delta_{\upsilon'} y^\upsilon = \delta^\upsilon_{\upsilon'} \quad (4.128)
$$

24. Demonstrate that if there is a scalar function $\tau(x)$ such that $\Delta_o\tau=1$, then we have the following commutation relation and implications for any arbitrary vector function $a_{\upsilon'}(\upsilon)$ of the proper-active strategies:

$$
\begin{aligned}
&\Delta_o\Delta_\upsilon\tau-\Delta_\upsilon\Delta_o\tau=-\left(q_\upsilon+2\theta_\upsilon{}^\alpha e_\alpha\right)\Rightarrow\Delta_o\left(\Delta_\upsilon\tau\right)=-\left(q_\upsilon+2\theta_\upsilon{}^\alpha e_\alpha\right) \\
&\Rightarrow\Delta_\upsilon\tau=2a_\upsilon-\lambda_\upsilon\tau \\
&\lambda_\upsilon=q_\upsilon+2\theta_\upsilon{}^\alpha e_\alpha
\end{aligned} \quad (4.129)
$$

25. Using exercise 24, show the following relationship between the differential operators Δ_o and Δ_υ with the partial derivatives ∂_τ and ∂_υ when operating on a scalar function $\varphi\left(y(x),\tau(x)\right)$:

$$
\begin{aligned}
\Delta_o\varphi&=\partial_\tau\varphi \\
\Delta_\upsilon\varphi&=\partial_\upsilon\varphi+\left(2a_\upsilon-\lambda_\upsilon\tau\right)\partial_\tau\varphi
\end{aligned} \quad (4.130)
$$

26. Starting with a power series expansion for the coordinate x^a in terms of the scalar function τ defined in exercise 24 and coefficients $X_n{}^a$ that are functions only of the scalar fields y^υ (which are scalar functions of the *harmonic coordinates*), use the commutation rule result (4.129) to show the expansions for the vector fields $E^a{}_o$ for the flow and $E^a{}_\upsilon$ for the active transverse directions, defining the scalar field $\lambda_\upsilon=q_\upsilon+2\theta_\upsilon{}^\alpha e_\alpha$:

$$
\begin{aligned}
x^a &= \sum_{n=0} X_n{}^a \tau^n \\
E^a{}_o &= \sum_{n=0} X_n{}^a n\tau^{n-1} \\
E^a{}_\upsilon &= \sum_{n=0} \partial_\upsilon X_n{}^a \tau^n - \sum_{n=0} \lambda_\upsilon X_n{}^a n\tau^n + \sum_{n=0} 2a_\upsilon X_n{}^a n\tau^{n-1}
\end{aligned}
\tag{4.131}
$$

27. Use exercise 20 to show the first commutation rule of Eq. (3.60) is satisfied:

$$
\Delta_\upsilon E^a{}_o - \Delta_o E^a{}_\upsilon = \Delta_\upsilon \Delta_o x^a - \Delta_o \Delta_\upsilon x^a = \left(q_\upsilon + 2\theta_\upsilon{}^\alpha e_\alpha\right) E^a{}_o \tag{4.132}
$$

28. Use exercise 20 with the scalar τ and the assumption that the scalar fields are static to show that a_υ is constrained by the magnetic and tidal magnetic fields and that the curl of λ_υ must be zero:

$$
\begin{aligned}
&\Delta_\upsilon \Delta_{\upsilon'} \tau - \Delta_{\upsilon'} \Delta_\upsilon \tau = 2\left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) \Delta_o \tau \\
&2\Delta_\upsilon a_{\upsilon'} - 2\Delta_{\upsilon'} a_\upsilon + 2\lambda_\upsilon a_{\upsilon'} - 2\lambda_{\upsilon'} a_\upsilon + \left(\Delta_{\upsilon'} \lambda_\upsilon - \Delta_\upsilon \lambda_{\upsilon'}\right)\tau = 2\left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) \\
&\Delta_\upsilon a_{\upsilon'} - \Delta_{\upsilon'} a_\upsilon + \lambda_\upsilon a_{\upsilon'} - \lambda_{\upsilon'} a_\upsilon = \omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'} \\
&\Delta_{\upsilon'} \lambda_\upsilon - \Delta_\upsilon \lambda_{\upsilon'} = 0
\end{aligned}
\tag{4.133}
$$

29. Use exercise 20 to show the second commutation rule of Eq. (3.60) is satisfied with the vector field a_υ satisfying Eq. (4.133):

$$
\Delta_\upsilon E^a{}_{\upsilon'} - \Delta_{\upsilon'} E^a{}_\upsilon = \Delta_\upsilon \Delta_{\upsilon'} x^a - \Delta_{\upsilon'} \Delta_\upsilon x^a = 2\left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) E^a{}_o \tag{4.134}
$$

30. Show that the co-moving coordinate basis field equations from section 4.4 for the *player fixed frame model* satisfy the *normal-form coordinate basis* longitudinal flows, Eq. (1.75) and (1.76), modified to include the inertial acceleration:

$$
\begin{aligned}
&g_{ab} \Delta_o V^b + \varpi_{abc} V^b V^c = V_k F^k_{ab} V^b - \tfrac{1}{2} V_j V_k \partial_a \gamma^{jk} + q_a \\
&\Delta_o V_k = q_k
\end{aligned}
\tag{4.135}
$$

31. Show that imposing a gauge condition, such as the following, on the vector field a_υ in Eq. (4.133) corresponds to a frame transformation, taking the most general form of λ_υ to be $\lambda_\upsilon = \partial_\upsilon q$ in terms of a scalar field:

$$
\partial_\upsilon \left(a^\upsilon \mathrm{e}^q\right) = 0 \tag{4.136}
$$

32. Show that the "curl" of the new ***characteristic vector potential*** field $-2\mathrm{e}^q a_\upsilon$ from Eq. (4.133) simplifies and determines the ***characteristic payoff*** and acceleration fields. Show that the field equations for the "curl" of the magnetic fields satisfy these equations and further, show that with the gauge from the previous exercise, the vector field depends on the ***player current*** in the second order differential equation shown below:

$$
\begin{aligned}
&\partial_\upsilon \left(\mathrm{e}^q a_{\upsilon'}\right) - \partial_{\upsilon'} \left(\mathrm{e}^q a_\upsilon\right) = \mathrm{e}^q \left(\omega_{\upsilon\upsilon'} - e_\alpha \omega^\alpha{}_{\upsilon\upsilon'}\right) \equiv \bar{f}_{\upsilon\upsilon'} \\
&\partial_\upsilon \bar{f}_{\upsilon'\upsilon''} + \partial_{\upsilon'} \bar{f}_{\upsilon''\upsilon} + \partial_{\upsilon''} \bar{f}_{\upsilon\upsilon'} = 0 \\
&-\mathrm{e}^{-q} \partial_{\upsilon'} \partial^{\upsilon'} \left(\mathrm{e}^q a_\upsilon\right) = \begin{pmatrix} \kappa e^\alpha p_{\alpha\upsilon} + \left(\omega_{\upsilon\upsilon'} - e^\alpha \omega_{\alpha\upsilon\upsilon'}\right)\left(\Theta^{\upsilon'} + q^{\upsilon'}\right) \\ +2\left(\omega_{\upsilon\upsilon'} e_\alpha - \omega_{\alpha\upsilon\upsilon'}\right)\omega^{\upsilon'\alpha} + 2\omega_{\upsilon\upsilon'} q^{\upsilon'} - 2\omega_{\alpha\upsilon\upsilon'} \omega^{\upsilon'\alpha\beta} e_\beta \end{pmatrix}
\end{aligned}
\tag{4.137}
$$

33. Verify Eq. (4.91) and (4.92). Use these equations to determine the general form of the power series in terms of the initial functions $X^a{}_0$ and $X^a{}_1$:

$$X_{n+2}{}^{a} = \frac{1}{(n+1)(n+2)\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right)} \begin{pmatrix} -\partial^\upsilon \partial_\upsilon X_n{}^a + \left((2n+1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_n{}^a \\ -n\left((n+1)q^\upsilon q_\upsilon + q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon\right) X_n{}^a \\ +2(n+1)\left(2(n+1)a_\upsilon q^\upsilon + a_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon a^\upsilon\right) X_{n+1}{}^a \\ -4(n+1)a^\upsilon \partial_\upsilon X_{n+1}{}^a \end{pmatrix} \quad (4.138)$$

34. Show that the partial differential equation corresponding to the wave equation Eq. (4.90) is:

$$\begin{pmatrix} \left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon - 4a^\upsilon q_\upsilon \tau + q^\upsilon q_\upsilon \tau^2\right)\partial_\tau{}^2 x^a + \partial_\upsilon \partial^\upsilon x^a + 2\left(2a^\upsilon - q^\upsilon \tau\right)\partial_\upsilon \partial_\tau x^a \\ +\left(-2\omega^{\upsilon\alpha}{}_\alpha a_\upsilon - 6a^\upsilon q_\upsilon + q_\upsilon \omega^{\upsilon\alpha}{}_\alpha \tau - \partial_\upsilon q^\upsilon \tau + 2q_\upsilon q^\upsilon \tau\right)\partial_\tau x^a - \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon x^a \end{pmatrix} = 0 \quad (4.139)$$

35. Use the power series solution to show that the wave equation Eq. (4.139) has polynomial solutions $P_N{}^a(\upsilon,\tau)$ with $X_{N+1}{}^a = X_{N+2}{}^a = 0$ and show that the order of solving for the coefficients starts with the equation for X_N and then equation for $X_{N-1}{}^a$ using this solution and so forth:

$$\partial_\upsilon \partial^\upsilon X_N{}^a - \left((2N+1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_N{}^a + N\left(q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + q^\upsilon q_\upsilon (N+1)\right) X_N{}^a = 0$$

$$\begin{pmatrix} \partial_\upsilon \partial^\upsilon X_{N-1}{}^a - \left((2N-1)q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon X_{N-1}{}^a + (N-1)\left(q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + Nq^\upsilon q_\upsilon\right) X_{N-1}{}^a \\ +4Na^\upsilon \partial_\upsilon X_N{}^a - 2N\left(\omega^{\upsilon\alpha}{}_\alpha a_\upsilon + (2N+1)a^\upsilon q_\upsilon\right) X_N{}^a \end{pmatrix} = 0$$

$$n = N-2,\cdots,0 \quad (4.140)$$

$$\begin{pmatrix} \partial_\upsilon \partial^\upsilon X_n{}^a - \left(\omega^{\upsilon\alpha}{}_\alpha + (2n+1)q^\upsilon\right)\partial_\upsilon X_n{}^a + n\left(q_\upsilon \omega^{\upsilon\alpha}{}_\alpha - \partial_\upsilon q^\upsilon + (n+1)q^\upsilon q_\upsilon\right) X_n{}^a \\ +4(n+1)a^\upsilon \partial_\upsilon X_{n+1}{}^a - 2(n+1)\left(\omega^{\upsilon\alpha}{}_\alpha + (2n+3)q^\upsilon\right) a_\upsilon X_{n+1}{}^a \\ +(n+2)(n+1)\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right) X_{n+2}{}^a \end{pmatrix} = 0$$

36. For the single active strategy streamline solutions of section 4.6, show that the inertial forces, Table 4-6, provides the diagonal components of stress, Eq. (4.35):

$$-\omega^{\upsilon\beta}{}_\beta q_\upsilon + \kappa\left(p^\alpha{}_\alpha - ph^\alpha{}_\alpha - \mu + \frac{\mu - p}{n-1}h^\alpha{}_\alpha\right) + \omega_{\upsilon\alpha}\omega^{\upsilon\alpha} + \tfrac{1}{2}\omega^{\upsilon\alpha}{}_\beta \omega_{\upsilon\alpha}{}^\beta - \tfrac{1}{2}\omega_{\upsilon\alpha}{}^\alpha \omega^{\upsilon\beta}{}_\beta = 0 \quad (4.141)$$

37. For the single active strategy streamline solutions of section 4.6, show that the inertial forces, Table 4-6, provide the divergence of the acceleration field given by Eq. (4.58).
38. For the single active strategy streamline solutions of section 4.6, show that Eq. (4.109) can be used to determine the divergence of the *bond compression*:

$$\partial_\upsilon \omega^{\upsilon\beta}{}_\beta = 3\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} + \tfrac{1}{2}\omega^{\alpha\upsilon}{}_\beta \omega^\beta{}_{\upsilon\alpha} + \tfrac{1}{2}\omega^\alpha{}_{\upsilon\alpha}\omega^{\beta\upsilon}{}_\beta - \kappa\mu \quad (4.142)$$

39. For the single active strategy streamline solutions of section 4.6, using the divergence of the acceleration Eq. (4.111), show that the *harmonic wave equation* Eq. (4.105) can be expressed as:

$$\begin{pmatrix} \left(1+e_\alpha e^\alpha + q^\upsilon q_\upsilon \tau^2\right)\partial_\tau{}^2 x^a + \partial_\upsilon \partial^\upsilon x^a - 2q^\upsilon \tau \partial_\upsilon \partial_\tau x^a \\ +\left(q_\upsilon q^\upsilon + 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \kappa\left(\mu + p - \frac{\mu - p}{n-1}\right)\right)\tau \partial_\tau x^a - \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\partial_\upsilon x^a \end{pmatrix} = 0 \quad (4.143)$$

40. For the single active strategy streamline solutions of section 4.6, using the divergence of the acceleration Eq. (4.111), show that the ***harmonic polynomials*** Eq. (4.106) can be expressed as:

$$
\begin{aligned}
&X_{N+1}{}^{a} = X_{N+2}{}^{a} = 0 \\
&\begin{pmatrix} \partial_{\upsilon}\partial^{\upsilon} X_{N}{}^{a} - \left(\left(2N+1\right)q^{\upsilon} + \omega^{\upsilon\alpha}{}_{\alpha}\right)\partial_{\upsilon} X_{N}{}^{a} \\ +N\left(Nq^{\upsilon}q_{\upsilon} + 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \kappa\left(\mu + p - \dfrac{\mu - p}{n-1}\right)\right) X_{N}{}^{a} \end{pmatrix} = 0 \\
&\begin{pmatrix} \partial_{\upsilon}\partial^{\upsilon} X_{N-1}{}^{a} - \left(\left(2N-1\right)q^{\upsilon} + \omega^{\upsilon\alpha}{}_{\alpha}\right)\partial_{\upsilon} X_{N-1}{}^{a} \\ +\left(N-1\right)\left(\left(N-1\right)q^{\upsilon}q_{\upsilon} + 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \kappa\left(\mu + p - \dfrac{\mu - p}{n-1}\right)\right) X_{N-1}{}^{a} \end{pmatrix} = 0 \\
&for: s = N-2, N-3, \cdots, 0 \\
&\begin{pmatrix} \partial_{\upsilon}\partial^{\upsilon} X_{s}{}^{a} + s\left(sq^{\upsilon}q_{\upsilon} + 2\omega_{\upsilon\alpha}\omega^{\upsilon\alpha} - \kappa\left(\mu + p - \dfrac{\mu - p}{n-1}\right)\right) X_{s}{}^{a} \\ -\left(\left(2s+1\right)q^{\upsilon} + \omega^{\upsilon\alpha}{}_{\alpha}\right)\partial_{\upsilon} X_{s}{}^{a} + \left(s+2\right)\left(s+1\right)\left(1 + e_{\alpha}e^{\alpha}\right) X_{s+2}{}^{a} \end{pmatrix} = 0
\end{aligned}
\tag{4.144}
$$

41. In exercise 25, we express the frame derivatives $\{\Delta_o \quad \Delta_\upsilon\}$ in terms of holonomic partial derivatives $\{\partial_\tau \quad \partial_\upsilon\}$. In effect we have made a frame transformation from the *co-moving orthonormal coordinate basis* to the ***central holonomic frame***. We include in the definition of *central holonomic frame* a redefinition of proper time $\overline{\tau} = \tau \mathrm{e}^{q}$ (which is just a scale change along a streamline). Show that the transformation $W^{\overline{\mu}}{}_{o} \quad W^{\overline{\mu}}{}_{\alpha} \quad W^{\overline{\mu}}{}_{\upsilon}$ from the *co-moving orthonormal frame* to the *central holonomic frame* has the following values for *central holonomic frame* coordinates $\overline{\mu} = \{\overline{\tau} \quad \overline{\upsilon} \quad \overline{k}\}$, where the first two coordinates are active and holonomic and the last coordinate is inactive and non-holonomic:

$W^{\overline{\tau}}{}_{o} = \mathrm{e}^{q}$	$W^{\overline{\tau}}{}_{\alpha} = \mathrm{e}^{q} e_{\alpha}$	$W^{\overline{\tau}}{}_{\upsilon} = 2a_{\upsilon}\,\mathrm{e}^{q}$	
$W^{\overline{\upsilon}}{}_{o} = 0$	$W^{\overline{\upsilon}}{}_{\alpha} = 0$	$W^{\overline{\upsilon}}{}_{\upsilon} = \delta^{\overline{\upsilon}}_{\upsilon}$	(4.145)
$W^{\overline{k}}{}_{o} = E^{\overline{k}}{}_{o}$	$W^{\overline{k}}{}_{\alpha} = E^{\overline{k}}{}_{\alpha}$	$W^{\overline{k}}{}_{\upsilon} = 0$	

42. Show that the transformation $\{W_{\overline{\mu}}{}^{o} \quad W_{\overline{\mu}}{}^{\alpha} \quad W_{\overline{\mu}}{}^{\upsilon}\}$ from the *central holonomic frame* to the *co-moving orthonormal frame* has the following values and is the inverse of the transformation Eq. (4.145):

$W_{\overline{\tau}}{}^{o} = \dfrac{\mathrm{e}^{-q}}{1+e_{\alpha}e^{\alpha}}$	$W_{\overline{\tau}}{}^{\alpha} = \dfrac{\mathrm{e}^{-q}\,e^{\alpha}}{1+e^{\beta}e_{\beta}}$	$W_{\overline{\tau}}{}^{\upsilon} = 0$	
$W_{\overline{\upsilon}}{}^{o} = \dfrac{-2a_{\overline{\upsilon}}}{1+e_{\alpha}e^{\alpha}}$	$W_{\overline{\upsilon}}{}^{\alpha} = \dfrac{-2a_{\overline{\upsilon}}}{1+e^{\beta}e_{\beta}}e^{\alpha}$	$W_{\overline{\upsilon}}{}^{\upsilon} = \delta^{\upsilon}_{\overline{\upsilon}}$	(4.146)
$W_{\overline{k}}{}^{o} = E_{\overline{k}}{}^{o}$	$W_{\overline{k}}{}^{\alpha} = E_{\overline{k}}{}^{\alpha}$	$W_{\overline{k}}{}^{\upsilon} = 0$	

43. Using the frame transformations from exercise 41, show that the following provide the frame transformation $\overline{E}^{\mu}_{\overline{\mu}}$ from the *normal-form coordinate basis* to the *central holonomic frame basis*. In addition show that the inverse frame transformation $\overline{E}_{\mu}{}^{\overline{\mu}}$ is as indicated below.

$$\begin{array}{|l:l|l|}
\hline
\bar{E}^{a}_{\bar{\tau}} = \mathrm{e}^{-q}\, E^{a}{}_{o} & \bar{E}^{a}_{\bar{\upsilon}} = E^{a}{}_{\bar{\upsilon}} - 2a_{\bar{\upsilon}} E^{a}{}_{o} & \bar{E}^{a}_{\bar{k}} = 0 \\
\hline
\bar{E}^{j}_{\bar{\tau}} = 0 & \bar{E}^{j}_{\bar{\upsilon}} = 0 & \bar{E}^{j}_{\bar{k}} = \delta^{j}_{\bar{k}} \\
\hline
\end{array}$$

$$\begin{array}{|l:l|l|}
\hline
\bar{E}_{a}{}^{\bar{\tau}} = \left(1 + e_{\alpha} e^{\alpha}\right) \mathrm{e}^{q}\, E_{a}{}^{o} + 2a_{\upsilon}\, \mathrm{e}^{q}\, E_{a}{}^{\upsilon} & \bar{E}_{a}{}^{\bar{\upsilon}} = E_{a}{}^{\bar{\upsilon}} & \bar{E}_{a}{}^{\bar{k}} = 0 \\
\hline
\bar{E}_{j}{}^{\bar{\tau}} = 0 & \bar{E}_{j}{}^{\bar{\upsilon}} = 0 & \bar{E}_{j}{}^{\bar{k}} = \delta^{\bar{k}}_{j} \\
\hline
\end{array} \qquad (4.147)$$

44. In the *central holonomic frame*, show that the inactive metric elements $\bar{\gamma}_{jk} = \gamma_{jk}$ are equal to the corresponding inactive metric elements in the *normal-form coordinate frame*, show that the mixed metric elements $\bar{g}^{\bar{k}b} = 0$ are zero and show that the active contravariant metric elements $\bar{g}^{\bar{a}\bar{b}}$ have the values given below. Furthermore, show that the determinant of the active metric elements are as indicated:

$$\begin{array}{|l|l|}
\hline
\bar{g}^{\bar{\tau}\bar{\tau}} = \left(1 + e_{\alpha} e^{\alpha} + 4a_{\upsilon} a^{\upsilon}\right) \mathrm{e}^{2q} & \bar{g}^{\bar{\tau}\bar{\upsilon}} = 2a^{\bar{\upsilon}}\, \mathrm{e}^{q} \\
\hline
\bar{g}^{\bar{\upsilon}\bar{\tau}} = 2a^{\bar{\upsilon}}\, \mathrm{e}^{q} & \bar{g}^{\bar{\upsilon}\bar{\upsilon}'} = -\delta^{\bar{\upsilon}\bar{\upsilon}'} \\
\hline
\end{array} \qquad (4.148)$$

$$\left|\det \bar{g}^{\bar{a}\bar{b}}\right| = \left(1 + e_{\alpha} e^{\alpha}\right) \mathrm{e}^{2q}$$

45. In the *central holonomic frame*, show that the active covariant metric elements are:

$$\begin{array}{|l|l|}
\hline
\bar{g}_{\bar{\tau}\bar{\tau}} = \dfrac{\mathrm{e}^{-2q}}{1 + e_{\alpha} e^{\alpha}} & \bar{g}_{\bar{\tau}\bar{\upsilon}} = -\dfrac{2a_{\bar{\upsilon}}\, \mathrm{e}^{-q}}{1 + e_{\alpha} e^{\alpha}} \\
\hline
\bar{g}_{\bar{\upsilon}\bar{\tau}} = -\dfrac{2a_{\bar{\upsilon}}\, \mathrm{e}^{-q}}{1 + e_{\alpha} e^{\alpha}} & \bar{g}_{\bar{\upsilon}\bar{\upsilon}'} = -\delta_{\bar{\upsilon}\bar{\upsilon}'} + \dfrac{4a_{\bar{\upsilon}} a_{\bar{\upsilon}'}}{1 + e_{\alpha} e^{\alpha}} \\
\hline
\end{array} \qquad (4.149)$$

46. Using the fact from exercise 45 that the metric elements in the *central holonomic frame* are independent of ***central time*** $\bar{\tau}$, show that central time is indeed ***central*** (Thomas G. H., 2006), *i.e.* it is inactive and commutes with all inactive strategies. Furthermore show that the curl of the slightly redefined ***characteristic potential*** $-2a_{\upsilon}\, \mathrm{e}^{q}$ determines the ***characteristic payoff*** $f_{\upsilon\upsilon'} = -2\bar{f}_{\upsilon\upsilon'} = -2\mathrm{e}^{q}\left(\omega_{\upsilon\upsilon'} - e_{\alpha}\omega^{\alpha}{}_{\upsilon\upsilon'}\right)$, *Cf.* Eq. (4.137).

47. Use the frame transformation to obtain the player payoff matrices in the *central holonomic frame*, considered as tensors in the space of time and *active* and *inactive* strategies:

$$\begin{aligned}
&\bar{F}^{j}_{\bar{\upsilon}\bar{\tau}} = \mathrm{e}^{-q}\, f^{j}{}_{\bar{\upsilon}o} \\
&\bar{F}^{j}_{\bar{\upsilon}\bar{\upsilon}'} = f^{j}{}_{\bar{\upsilon}\bar{\upsilon}'} + 2a_{\bar{\upsilon}} f^{j}{}_{\bar{\upsilon}'o} - 2a_{\bar{\upsilon}'} f^{j}{}_{\bar{\upsilon}o} \\
&\bar{F}^{j}_{\bar{\tau}\bar{k}} = \bar{F}^{j}_{\bar{\upsilon}\bar{k}} = \bar{F}^{j}_{\bar{i}\bar{k}} = 0
\end{aligned} \qquad (4.150)$$

48. Based on the form of the payoff matrix Eq. (4.150) in the *central holonomic frame* and using the definition Eq. (3.30) transformed to the *central holonomic frame*, show that the payoff matrix for the active components is the curl $\bar{F}^{j}_{\bar{a}\bar{b}} = \partial_{\bar{a}} \bar{A}^{j}_{\bar{b}} - \partial_{\bar{b}} \bar{A}^{j}_{\bar{a}}$ of some potential function $\bar{A}^{j}_{\bar{a}}$ by showing:

$$\partial_{\bar{a}} \bar{F}^{j}_{\bar{b}\bar{c}} + \partial_{\bar{b}} \bar{F}^{j}_{\bar{c}\bar{a}} + \partial_{\bar{c}} \bar{F}^{j}_{\bar{a}\bar{b}} = 0 \qquad (4.151)$$

49. To determine the potential function $\bar{A}^{j}_{\bar{a}}$ implied by exercise 48, both the curl and divergence need to be specified. Show that the gauge condition $g^{ab}\partial_{a} A^{j}{}_{b} = 0$ in the normal form coordinate frame, which we call the ***harmonic gauge*** for the vector potential (Thomas G. H., 2006) , can be expressed in the *central holonomic frame* by establishing the following relations, where the covariant derivatives are here restricted to active components only:

$$\begin{aligned}
&\frac{1}{\sqrt{g\gamma}}\partial_b\left(\sqrt{g\gamma}g^{ab}A^j{}_a\right)=g^{ab}\partial_b A^j{}_a=0\\
&g^{ab}A^j{}_{a;b}=\frac{1}{\sqrt{|g|}}\partial_b\left(g^{ab}\sqrt{|g|}A^j{}_a\right)\\
&g^{ab}A^j{}_{a;b}+\tfrac{1}{2}g^{ab}A^j{}_a\partial_b\ln|\gamma|=0\\
&\frac{1}{\sqrt{g\gamma}}\partial_{\bar b}\left(\sqrt{g\gamma}\bar g^{\bar a\bar b}\bar A^j_{\bar a}\right)=\bar g^{\bar a\bar b}\partial_{\bar b}\bar A^j_{\bar a}+\frac{1}{\sqrt{g\gamma}}\partial_{\bar b}\left(\sqrt{g\gamma}\bar g^{\bar a\bar b}\right)\bar A^j_{\bar a}=0
\end{aligned}\tag{4.152}$$

50. Show that in the *central holonomic frame* the active flow components are $\left\{\bar V^{\bar\tau}=W^{\bar\tau}{}_o=e^q,\quad \bar V^{\bar\upsilon}=W^{\bar\upsilon}{}_o=0\right\}$ and the inactive flow components are $\bar V^{\bar k}=W^{\bar k}{}_o=V^{\bar k}$. If we call the covariant-flows the "momenta", show that these momenta are $\left\{\bar V_{\bar\tau}=\dfrac{\mathrm{e}^{-q}}{1+e^\alpha e_\alpha},\quad \bar V_{\bar\upsilon}=-\dfrac{2a_{\bar\upsilon}}{1+e_\alpha e^\alpha}\right\}$ for the active components and $\bar V_{\bar k}=V_{\bar k}$ for the inactive components. This shows that $a_{\bar\upsilon}$ contributes to the momentum.

51. Show that in the *central holonomic frame* the acceleration components are $\left\{\bar q^{\bar\tau}=2q_{\bar\upsilon}a^{\bar\upsilon}\,\mathrm{e}^q,\quad \bar q^{\bar\upsilon}=q^{\bar\upsilon}\right\}$. The covariant components (the "forces") are $\left\{\bar q_{\bar\tau}=0,\quad \bar q_{\bar\upsilon}=q_{\bar\upsilon},\quad \bar q_{\bar k}=0\right\}$. These acceleration components are for the full geometry of active and inactive coordinates. We can define the ***active geometry acceleration*** in frame in which the active coordinates are holonomic and in which the inactive coordinates are orthogonal using only the active acceleration, illustrated for the *central holonomic frame* in the first equation below. Show that in that frame, the *active geometry acceleration* consists of three contributions, shown in the second equation below. The first contribution is the ***competitive acceleration***, the second is the ***cooperative acceleration*** and the third is the ***absolute acceleration***. Show that the *active geometry acceleration* in the *central holonomic frame* is given by the third and fourth equations. Also show the identity of the *active geometry acceleration* as computed from the first equation and computed from the sum of the three contributions. In the process show that the *competitive acceleration* and *cooperative acceleration* have the forms provided in the last two equations. You will also have demonstrated that the three acceleration contributions are each independent of proper time as is the *active geometry acceleration* in the *central holonomic frame*.

$$\begin{aligned}
&\bar Q_{\bar a}=\bar g_{\bar a\bar b}\bar V^{\bar c}\partial_{\bar c}\bar V^{\bar b}+\bar\varpi_{\bar a\bar b\bar c}\bar V^{\bar b}\bar V^{\bar c}\\
&\bar Q_{\bar a}=V_{\bar k}\bar F^{\bar k}_{\bar a\bar b}\bar V^b+\tfrac{1}{2}V^{\bar j}V^{\bar k}\partial_{\bar a}\gamma_{\bar j\bar k}+\bar q_{\bar a}\\
&\bar Q_{\bar\tau}=0\\
&\bar Q_{\bar\upsilon}=\frac{q_{\bar\upsilon}+2\omega_{\bar\upsilon\alpha}e^\alpha+\omega_{\bar\upsilon\alpha\beta}e^\alpha e^\beta}{\left(1+e_\alpha e^\alpha\right)^2}\\
&E_{ko}f^k_{\bar\upsilon o}=\frac{-2q_{\bar\upsilon}e_\gamma e^\gamma-2\left(e_\gamma e^\gamma-1\right)\omega_{\bar\upsilon\alpha}e^\alpha+2\omega_{\bar\upsilon\alpha\beta}e^\alpha e^\beta}{\left(1+e_\gamma e^\gamma\right)^2}\\
&\tfrac{1}{2}E^l{}_oE^m{}_o\partial_{\bar\upsilon}\gamma_{lm}=\frac{-q_{\bar\upsilon}e_\gamma e^\gamma e_\beta e^\beta+2\omega_{\bar\upsilon\alpha}e^\alpha e_\gamma e^\gamma-\omega_{\bar\upsilon\alpha\beta}e^\alpha e^\beta}{\left(1+e_\gamma e^\gamma\right)^2}
\end{aligned}\tag{4.153}$$

52. Using exercise 51 for the *active geometry acceleration* in the *central holonomic frame* and the transformation properties exercise 43, show that the *active geometry acceleration* in the *normal-*

form coordinate basis is $Q_a = E_a{}^{\bar{\upsilon}} \bar{Q}_{\bar{\upsilon}}$. So in particular the time dependence is provided entirely by the transformation $E_a{}^{\bar{\upsilon}}$. Furthermore, show that the covariant magnitude of this acceleration $g^{ab} Q_a Q_b$ is $-\delta^{\bar{\upsilon}\bar{\upsilon}'} \bar{Q}_{\bar{\upsilon}} \bar{Q}_{\bar{\upsilon}'}$ and so a scalar independent of proper time: like the pressure p, energy density μ and cooperation potentials γ_{jk}, the covariant magnitude is constant along a streamline. Show that for any scalar φ that is constant along the streamline, the gradient is $\partial_a \varphi = E_a{}^{\bar{\upsilon}} \partial_{\bar{\upsilon}} \varphi$ and so show that in the *normal-form coordinate basis* such scalars will in general be time dependent if there are non-trivial *harmonics*.

53. In our initial numerical analysis (Thomas G. H., 2006), in addition to an exactly solved single strategy model, we computed a variety of examples in which all strategies were *active*. In these models, we assumed that there was a ***central holonomic frame*** in which time was inactive and commuted with all active strategies. For the *player fixed frame model* with the *quasi-stationary hypothesis*, we satisfy that assumption (exercise 46). We also made several practical simplifications to evaluate these models: for example we did not use the field equations to obtain the metric and payoff tensors, but estimated their behaviors; we averaged the *player payoffs*, which set the value of the *characteristic payoff* since we assumed the *composite payoff* was zero and the frame was fixed; and we assumed that the relationship between energy density and pressure was that of a perfect fluid. In this chapter with the *player fixed frame model* with the *quasi-stationary hypothesis*, we have removed those simplifications and estimates and provided exact solutions. In particular we have explicitly kept each player's *inactive strategy* (no common inactive strategy) and don't assume that the *composite payoff* is zero. Show that the *normal-form coordinate basis* behaviors seen from that initial numerical analysis now follow directly from the ***free fall behavior*** of $E^a{}_o$ (*Cf.* section 11.5). Show that the gradient of the pressure and the gradient of the cooperation potentials γ_{jk} are computed from $E_a{}^{\bar{\upsilon}}$ (exercise 52).
54. Show that the *wave equation* Eq. (4.90) in the *central holonomic frame* is transformed from the partial differential equation Eq. (4.94) to the following partial differential equation in which the coefficients are all independent of the central time $\bar{\tau}$:
$$\begin{pmatrix} \left(1 + e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right) \mathrm{e}^{2q}\, \bar{\partial}_{\bar{\tau}}{}^2 x^a - \delta^{\upsilon\upsilon'} \bar{\partial}_{\upsilon'} \bar{\partial}_\upsilon x^a \\ +4a^\upsilon\, \mathrm{e}^q\, \bar{\partial}_\upsilon \bar{\partial}_{\bar{\tau}} x^a - 2\,\mathrm{e}^q\, a_\upsilon \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) \bar{\partial}_{\bar{\tau}} x^a - \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) \bar{\partial}_\upsilon x^a \end{pmatrix} = 0 \qquad (4.154)$$
55. Show that one can apply the phasor approach directly to the partial differential equation in exercise 54, namely one can superpose solutions of the type $x^a = x_\omega{}^a\, \mathrm{e}^{i\varpi\bar{\tau}}$ where the phasor component $x_\omega{}^a$ satisfies the following equation that is independent of the *central time*:
$$\begin{pmatrix} \delta^{\upsilon\upsilon'} \bar{\partial}_{\upsilon'} \bar{\partial}_\upsilon x_\varpi{}^a + \left(\varpi^2\, \mathrm{e}^{2q} \left(1 + e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right) + 2i\varpi\, \mathrm{e}^q\, a_\upsilon \left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon\right)\right) x_\varpi{}^a \\ + \left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon - 4i\varpi\, \mathrm{e}^q\, a^\upsilon\right) \bar{\partial}_\upsilon x_\varpi{}^a \end{pmatrix} = 0 \qquad (4.155)$$
56. Show that the equation in exercise 54 also has linear solutions:
$$\begin{aligned} & x^a = U_0{}^a + V_0{}^a \bar{\tau} \\ & \delta^{\upsilon\upsilon'} \bar{\partial}_{\upsilon'} \bar{\partial}_\upsilon U_0{}^a + \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) \bar{\partial}_\upsilon U_0{}^a + 2\,\mathrm{e}^q\, a_\upsilon \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) V_0{}^a - 4a^\upsilon\, \mathrm{e}^q\, \bar{\partial}_\upsilon V_0{}^a = 0 \qquad (4.156) \\ & \delta^{\upsilon\upsilon'} \bar{\partial}_{\upsilon'} \bar{\partial}_\upsilon V_0{}^a + \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right) \bar{\partial}_\upsilon V_0{}^a = 0 \end{aligned}$$
57. The solutions in exercise 55 of the wave equation are in terms of complex numbers. Show that the solutions can also be written in terms of sines and cosines as the two coupled sets of differential equations below. Note that the freedom to choose the coefficients rests entirely in specifying the boundary conditions. Once the boundary conditions are set, the remaining behaviors are coupled as shown.

$$x^a = U_\varpi{}^a \cos\varpi\tau + V_\varpi{}^a \sin\varpi\tau$$

$$\begin{pmatrix} \delta^{\upsilon\upsilon'}\bar{\partial}_{\upsilon'}\bar{\partial}_\upsilon U_\varpi{}^a + \varpi^2\, \mathrm{e}^{2q}\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right)U_\varpi{}^a + \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\bar{\partial}_\upsilon U_\varpi{}^a \\ +2\varpi\, \mathrm{e}^q\, a_\upsilon\left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)V_\varpi{}^a - 4\varpi\, \mathrm{e}^q\, a^\upsilon \bar{\partial}_\upsilon V_\varpi{}^a \end{pmatrix} = 0 \qquad (4.157)$$

$$\begin{pmatrix} \delta^{\upsilon\upsilon'}\bar{\partial}_{\upsilon'}\bar{\partial}_\upsilon V_\varpi{}^a + \varpi^2\, \mathrm{e}^{2q}\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right)V_\varpi{}^a + \left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)\bar{\partial}_\upsilon V_\varpi{}^a \\ -2\varpi\, \mathrm{e}^q\, a_\upsilon\left(q^\upsilon + \omega^{\upsilon\alpha}{}_\alpha\right)U_\varpi{}^a + 4\varpi\, \mathrm{e}^q\, a^\upsilon \bar{\partial}_\upsilon U_\varpi{}^a \end{pmatrix} = 0$$

58. An estimate of the behavior of the solutions to the phasor equations is that the coefficients are constants in exercise 55 and the phasor solution is a wave $x_\varpi{}^a = \mathrm{e}^{k_\upsilon y^\upsilon}$. The values for example could be the initial values and the assumption is that they change slowly with position. Show that the propagation vector k_υ satisfies the following algebraic equation. Discuss solutions to this equation if the propagation vector were entirely along a single axis.

$$-k_\upsilon k^\upsilon + \left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon - 4i\varpi\, \mathrm{e}^q\, a^\upsilon\right)k_\upsilon + \varpi^2\, \mathrm{e}^{2q}\left(1+e_\alpha e^\alpha + 4a^\upsilon a_\upsilon\right) + 2i\varpi\, \mathrm{e}^q\, a_\upsilon\left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon\right) = 0 \quad (4.158)$$

59. As a crude estimate to the solution for the propagation vector in exercise 58, assume that the initial conditions are that $q^\upsilon = a^\upsilon = 0$ and that the propagation vector is along a single direction z with magnitude $\gamma = \sqrt{-k_\upsilon k^\upsilon}$. Show that the equation and solution are below. Infer that for sufficiently small frequencies there are two solutions to the wave equation, which have no oscillations and are attenuated along the $+z$ direction or along the $-z$ direction. Show that for sufficiently high frequencies there will be a travelling wave, but one that is still attenuated in general. Write the propagation magnitude as $\gamma = \alpha + i\beta$ and show that the velocity of the wave is ϖ/β. What is the velocity of the wave for sufficiently large frequencies? The source of the attenuation in both cases is the presence of a non-zero compression $\omega^{z\alpha}{}_\alpha$. Discuss whether these insights will carry over to solutions that take into account the spatial dependences.

$$\gamma^2 + \omega^{z\alpha}{}_\alpha \gamma + \varpi^2\left(1+e_\alpha e^\alpha\right) = 0 \Rightarrow \gamma = \frac{-\omega^{z\alpha}{}_\alpha \pm \sqrt{\omega^{z\alpha}{}_\alpha \omega^{z\alpha}{}_\alpha - 4\varpi^2\left(1+e_\alpha e^\alpha\right)}}{2} \qquad (4.159)$$

60. For the single strategy model, show that the *wave equation* (exercise 54) simplifies to:

$$\delta^{\upsilon\upsilon'}\bar{\partial}_{\upsilon'}\bar{\partial}_\upsilon x_\varpi{}^a + \varpi^2\, \mathrm{e}^{2q}\left(1+e_\alpha e^\alpha\right)x_\varpi{}^a + \left(\omega^{\upsilon\alpha}{}_\alpha + q^\upsilon\right)\bar{\partial}_\upsilon x_\varpi{}^a = 0 \qquad (4.160)$$

61. In the *central holonomic frame* the *player payoff fields* are determined by a current, Eq. (3.14). Show that the current components are given in terms of the following expressions involving the *co-moving orthonormal* scalars:

$$\begin{aligned} \bar{p}_j{}^{\bar{\upsilon}} &= E_j{}^\alpha p_\alpha{}^{\bar{\upsilon}} \\ \bar{p}_j{}^{\bar{\tau}} &= \mathrm{e}^q\, E_j{}^\alpha p_{\alpha\beta} e^\beta + 2\,\mathrm{e}^q\, E_j{}^\alpha a_\upsilon p_\alpha{}^\upsilon \end{aligned} \qquad (4.161)$$

62. Show that the gauge condition implicitly chosen in the *central holonomic frame* is

$$\partial_{\bar{b}}\bar{g}^{\bar{a}\bar{b}} = 0 \qquad (4.162)$$

63. Assume that there is a Killing vector $K^\mu = \phi V^\mu$ proportional to the flow V^μ that commutes with all the inactive central strategies, those that mutually commute with each other. Show that these mutually commuting set of Killing vectors, along with the remaining active coordinates form coordinate potentials that can be defined for all coordinates and form the *central holonomic frame*. Show that this frame is co-moving, but not orthonormal in general. Transform to a new frame, the ***active-co-moving frame***, in which only the active spatial flow components are zero using the transformation below and show that the form of the line element is as stated below.

Structurally there is no difference in form between the *central holonomic frame* or the *active-co-moving frame*.

$$\begin{aligned}
&W^{\alpha}{}_{\mu} = \begin{pmatrix} \delta^j_k & \phi V^j \delta^\tau_b \\ 0 & \delta^a_b \end{pmatrix} \\
&\left(W^{-1}\right)^{\mu}{}_{\alpha} = \begin{pmatrix} \delta^j_k & -\phi V^j \delta^\tau_b \\ 0 & \delta^a_b \end{pmatrix} \\
&\boldsymbol{\Xi}^j = d\xi^j + \phi V^j \delta^\tau_a dx^a \\
&\bar{A}^k_a = \hat{A}^k_a - \phi V^k \delta^\tau_a \\
&\mathbf{U}^j = d\xi^j + \hat{A}^j_a dx^a = \boldsymbol{\Xi}^j + \bar{A}^j_a d\bar{x}^a \\
&ds^2 = \hat{\gamma}_{jk}\mathbf{U}^j\mathbf{U}^k + \hat{g}_{ab}dx^a dx^b = \hat{\gamma}_{jk}\mathbf{U}^j\mathbf{U}^k + \hat{g}_{ab}d\bar{x}^a d\bar{x}^b
\end{aligned} \tag{4.163}$$

64. Show that the choice of gauge does not impact physical quantities. Thus the curvature tensor and energy momentum tensor are not changed by the gauge choice. If they display time dependence then the time dependence is gauge independent. In particular, show that if there is a time like Killing vector proportional to the flow, then if we solve the equations for the metric in the *active-co-moving frame* with the gauge Eq. (4.162), we can then use Eq. (4.156) to obtain the solution of the initial problem in the harmonic gauge. In this case show that the curvature and energy momentum tensors will be *stationary*.
65. Articulate the ***central frame stationary hypothesis*** as the statement that in the *central holonomic frame* in which the active strategies are holonomic and the active metric components are orthogonal to the inactive metric components, one of the central inactive metric components is time. By constructing the ***stationary orthonormal coordinate basis*** that starts with the central time, show that the orientation potentials in this basis are stationary. Show that in general this basis will not be the same as the *co-moving orthonormal coordinate basis*. Show that the key difference occurs when the *central holonomic frame* active flow is not zero. Show that there is a transformation from the *normal-form coordinate basis* to the *central-form coordinate basis* that respects the holonomic character of the active components and transforms the inactive and active components independently. Use this to show that we may take the inactive components to be the same in the two bases. The transformation from the *central holonomic frame coordinate basis* in which the metric is independent of time to the *normal-form coordinate basis* can be accomplished with the harmonic gauge transformation Eq. (3.13) written in the *central holonomic frame,* $\bar{\gamma}^{\mu\nu} x^a{}_{;\mu\nu} = 0$. Conclude that the most general model assuming the *central frame stationary hypothesis* is one that starts in the *central holonomic frame* for a *stationary* metric and solves the associated field equations for that (active strategy) holonomic system. Eq. (4.154) is a special case of this.
66. We have a complete set of equations that appear to have one additional constraint that has not been dealt with, Eq. (4.53). Show (or disprove) that this equation is identically satisfied as a consequence of the field equations.
67. It is a standard result that for the ***ownership model***, Eqs. (4.79) can be solved in terms of a potential, $p_{\upsilon\alpha} = \partial_\upsilon p_\alpha$. Show that this is also true for the ***conductivity model***, with the addition of an integrating factor. Show that the integrating factor can be obtained in terms of the acceleration potential and the transverse compression potential. Show that the gradient term is modified as shown. You will have to make use of the fact that the compression matrix commutes with its gradient to show that there is a single matrix integrating factor that works for all stresses $p_{\upsilon\alpha}$. In the matrix form of the equation (here we use bold to indicate a matrix), show that the form is as shown.

$$
\begin{aligned}
&p_{\upsilon\alpha} = \lambda_{P\alpha}{}^{\beta}\partial_{\upsilon}p_{\beta} \\
&\partial_{\upsilon}\lambda_{P\alpha}{}^{\beta} = \partial_{\upsilon}\left(q\delta_{\alpha}{}^{\gamma} - \varphi_{\alpha}{}^{\gamma}\right)\lambda_{P\gamma}{}^{\beta} \\
&\boldsymbol{\lambda}_{\mathbf{P}} = \exp\left(q\mathbf{I} - \boldsymbol{\varphi}\right) \\
&\partial_{\upsilon}\partial^{\upsilon}p_{\alpha} = \left(\partial_{\upsilon}\Theta\delta_{\alpha}{}^{\gamma} + 2\partial_{\upsilon}\varphi_{\alpha}{}^{\gamma}\right)\partial^{\upsilon}p_{\gamma}
\end{aligned}
\qquad (4.164)
$$

68. Assume we start with the potential $\overline{A}_{\upsilon} = \mathrm{e}^{q}a_{\upsilon}$ and its equation of motion Eq. (4.137). Assume that further we have the gauge conditions $\overline{\lambda} = \partial^{\upsilon}\overline{A}_{\upsilon} = 0$ and $\partial_{\upsilon}\overline{\lambda} = 0$. Show using the equations of motion from the full set of field equations that we have $-\partial^{\upsilon'}\partial_{\upsilon'}\overline{\lambda} = 0$, and so if the gauge conditions are satisfied on a surface they are satisfied everywhere. That means we only have to solve the equations for the potential:

$$
-\partial_{\upsilon'}\partial^{\upsilon'}\overline{A}_{\upsilon} = \mathrm{e}^{q}\begin{pmatrix} \kappa e^{\alpha}p_{\alpha\upsilon} + \left(\omega_{\upsilon\upsilon'} - e^{\alpha}\omega_{\alpha\upsilon\upsilon'}\right)\left(\Theta^{\upsilon'} + q^{\upsilon'}\right) \\ +2\left(\omega_{\upsilon\upsilon'}e_{\alpha} - \omega_{\alpha\upsilon\upsilon'}\right)\omega^{\upsilon'\alpha} + 2\omega_{\upsilon\upsilon'}q^{\upsilon'} - 2\omega_{\alpha\upsilon\upsilon'}\omega^{\upsilon'\alpha\beta}e_{\beta} \end{pmatrix} \qquad (4.165)
$$

69. Assume that we have the assumptions stated in exercise 68 and that z is a coordinate direction that is constant on the initial surface. Show that the gauge condition $\partial_{z}\overline{\lambda} = 0$ and the equations of motion leads to the following equations, which form an elliptic partial differential equation on the initial surface.

$$
\begin{aligned}
&\sum_{\upsilon}\partial_{\upsilon}\partial_{z}\overline{A}_{\upsilon} = 0 \Rightarrow \partial_{z}\partial_{z}\overline{A}_{z} + \sum_{\upsilon\neq z}\partial_{\upsilon}\partial_{z}\overline{A}_{\upsilon} = 0 \\
&\sum_{\upsilon}\partial_{\upsilon}\partial_{\upsilon}\overline{A}_{z} = \sum_{\upsilon\neq z}\left(\partial_{\upsilon}\partial_{\upsilon}\overline{A}_{z} - \partial_{\upsilon}\partial_{z}\overline{A}_{\upsilon}\right) \qquad (4.166) \\
&\sum_{\upsilon\neq z}\left(\partial_{\upsilon}\partial_{\upsilon}\overline{A}_{z} - \partial_{\upsilon}\partial_{z}\overline{A}_{\upsilon}\right) = \mathrm{e}^{q}\begin{pmatrix} \kappa e^{\alpha}p_{\alpha z} + \left(\omega_{z\upsilon'} - e^{\alpha}\omega_{\alpha z\upsilon'}\right)\left(\Theta^{\upsilon'} + q^{\upsilon'}\right) \\ +2\left(\omega_{z\upsilon'}e_{\alpha} - \omega_{\alpha z\upsilon'}\right)\omega^{\upsilon'\alpha} + 2\omega_{z\upsilon'}q^{\upsilon'} - 2\omega_{\alpha z\upsilon'}\omega^{\upsilon'\alpha\beta}e_{\beta} \end{pmatrix}
\end{aligned}
$$

70. The technique of using an integrating factor can also be applied to the equations of motion for Eq. (4.30), so that for any set of vector potentials $b_{\beta\upsilon}$ and for the matrix function $\boldsymbol{\lambda}_{\mathbf{B}}$ determined below, this equation is always satisfied.

$$
\begin{aligned}
&\left(\partial_{\upsilon}\omega_{\alpha\upsilon'\upsilon''} - \omega_{\alpha\upsilon\beta}\omega^{\beta}{}_{\upsilon'\upsilon''}\right)dx^{\upsilon} \wedge dx^{\upsilon'} \wedge dx^{\upsilon''} = 0 \\
&\omega_{\alpha\upsilon\upsilon'} = \lambda_{B\alpha}{}^{\beta}\left(\partial_{\upsilon}b_{\beta\upsilon'} - \partial_{\upsilon'}b_{\beta\upsilon}\right) \\
&\left(\partial_{\upsilon}\lambda_{B\alpha}{}^{\beta} + \partial_{\upsilon}\varphi_{\alpha}{}^{\gamma}\lambda_{B\gamma}{}^{\beta}\right)\left(\partial_{\upsilon'}b_{\beta\upsilon''} - \partial_{\upsilon''}b_{\beta\upsilon'}\right)dx^{\upsilon} \wedge dx^{\upsilon'} \wedge dx^{\upsilon''} = 0 \qquad (4.167) \\
&\partial_{\upsilon}\lambda_{B\alpha}{}^{\beta} + \partial_{\upsilon}\varphi_{\alpha}{}^{\gamma}\lambda_{B\gamma}{}^{\beta} = 0 \Rightarrow \boldsymbol{\lambda}_{\mathbf{B}} = \exp\left(-\boldsymbol{\varphi}\right) \\
&\partial^{\upsilon'}\omega_{\alpha\upsilon\upsilon'} = \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} - 2\omega^{\upsilon'}{}_{\alpha}\omega_{\upsilon\upsilon'} + \omega^{\upsilon'}{}_{\alpha\beta}\omega^{\beta}{}_{\upsilon\upsilon'} + \omega_{\upsilon'\beta}{}^{\beta}\omega_{\alpha\upsilon}{}^{\upsilon'} - \kappa p_{\alpha\upsilon}
\end{aligned}
$$

71. Continuing with the vector potentials from exercise 70, show that the equations of motion Eq. (4.77) can be determined with gauge conditions $\Psi_{\alpha} = \partial^{\upsilon}b_{\alpha\upsilon} = 0$ and $\partial_{\upsilon}\Psi_{\alpha} = 0$ defined on the surface:

$$
\begin{aligned}
&\partial^{\upsilon'}\omega_{\alpha\upsilon\upsilon'} = -\kappa p_{\alpha\upsilon} - 2\omega^{\upsilon'}{}_{\alpha}\omega_{\upsilon\upsilon'} + \omega_{\alpha\upsilon\upsilon'}q^{\upsilon'} + \Theta^{\upsilon'}\omega_{\alpha\upsilon\upsilon'} + \omega^{\upsilon'}{}_{\alpha\beta}\omega^{\beta}{}_{\upsilon\upsilon'} \Rightarrow \\
&-\partial_{\upsilon'}\partial^{\upsilon'}b_{\beta\upsilon} = \begin{pmatrix} -\lambda_{B}{}^{-1}{}_{\beta}{}^{\gamma}\left(\kappa p_{\gamma\upsilon} + 2\omega^{\upsilon'}{}_{\gamma}\omega_{\upsilon\upsilon'}\right) \\ +\left(q^{\upsilon'} + \Theta^{\upsilon'}\right)\left(\partial_{\upsilon}b_{\beta\upsilon'} - \partial_{\upsilon'}b_{\beta\upsilon}\right) + 2\omega^{\upsilon'}{}_{\beta\gamma}\left(\partial_{\upsilon}b^{\gamma}{}_{\upsilon'} - \partial_{\upsilon'}b^{\gamma}{}_{\beta\upsilon}\right) \end{pmatrix} \qquad (4.168)
\end{aligned}
$$

72. Show that for exercise 71 and an initial surface defined by a coordinate axis z being constant, the elliptic partial differential equations on the surface for the gauge condition are:

$$\sum_{\upsilon \neq z} \left(\partial_\upsilon \partial_\upsilon b_{\alpha z} - \partial_\upsilon \partial_z b_{\alpha \upsilon} \right) = \begin{pmatrix} -\lambda_B{}^{-1}{}_\beta{}^\gamma \left(\kappa p_{\gamma z} + 2\omega^{\upsilon'}{}_\gamma \omega_{z\upsilon'} \right) \\ + \left(\left(q^{\upsilon'} + \Theta^{\upsilon'} \right) h_{\beta\gamma} + 2\omega^{\upsilon'}{}_{\beta\gamma} \right) \left(\partial_z b^\gamma{}_{\upsilon'} - \partial_{\upsilon'} b^\gamma{}_{\beta z} \right) \end{pmatrix} \quad (4.169)$$

73. Discuss the significance of the source term and using exercise 67, show that the passion source term is a scalar multiple of the gradient $\partial_\upsilon p_\beta$. So in a sense, this ***conductivity model*** gradient is as related to ownership as is the one in the ***ownership model***.