The dynamics of decision processes: geometry, language and strategy

Gerald H Thomas

Milwaukee School of Engineering

October, 2012

A-1. Surface geometry of an oblate spheroid

Introduction

Several years ago around March, 1989, Bob Sargent, a good friend of mine, suggested the following problem that occurs in racing sail boats on Lake Michigan: accurately find the shortest distance between two cities on Lake Michigan. This is relevant to the handicapping system used in sail boat racing. The handicapping system is designed so that boats that race against each other that are different, for example in size, manufacturer, rigging and/or sails, can be ranked fairly. One such handicapping system was based entirely on the exact shortest distance between the start and finish of the race, irrespective of the course actually sailed by the boats. So for example a race between Chicago and Michigan City would need the exact distance between these two cities.

A somewhat empirical formula was then used to rank boats based on this exact distance. Even a tenth or one hundredth of a mile difference might make a difference in which boat was ranked first in a race. Typically, the organizers of the race would assume that the earth is flat and would use the appropriate charts for the Lake Michigan area for computing the distance. The charts assume that the earth is approximately flat in the area of interest. Bob raised the concern that according to the rules of racing, a boat could protest the result because the earth deviates from flat over distances of 20-30 miles in a way that might be significant to the boats in a race because of the handicapping system. In fact, Bob worried that a sufficiently knowledgeable sailor would recognize that the earth is in fact an oblate spheroid, whose characteristics are documented by the National Bureau of Standards. Since Bob knew this from his time in the Navy, he felt that his concern was well founded and calculations were needed to determine the size of the effect.

I was able to answer his question and in the process was forced to research and figure out how distances were computed on surfaces with different geometries. In the end, the answer to Bob's question was that the spherical nature of the earth is important, but the oblateness of the earth is probably not. In the process, I found the following interesting picture for a sphere with exaggerated flattening, indicating that the geodesics in general are not great circles but look like those in Figure 1-1.

The goal of this white paper is to indicate how this comes about and in the process learn something about finding the shortest paths on curved surfaces. In the process what I learn ties into research and work that applies these ideas to how we make strategic decisions (Thomas, 2006).

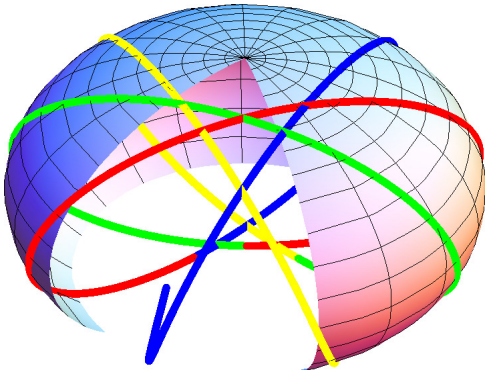

Figure 1-1: Oblate Spheroid

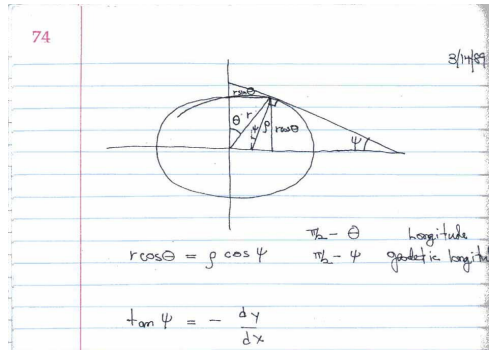

Figure 1-2: Geodetic polar angle

Mercator projection

Here is a copy, Figure 1-2, from my journal from 3/14/1989 with the diagram people have used to describe positions on the earth. It defines the ***geodetic polar angle*** ψ. I found interesting the following from the Oxford English Dictionary (Oxford University, 2009) on the origin of the word geodetic, coming from geodesy:

> " [ad. F. géodésie, ad. mod.L. geōdæsia, Gr. γεωδαισία, f. γεω-, γῆ earth + δαίειν to divide.] …
>
> †a.a Land surveying; the measuring of land (obs.). b.b In mod. use: That branch of applied mathematics which determines the figures and areas of large portions of the earth's surface, and the figure of the earth as a whole.
>
> 1570 Dee Math. Pref. 16 Of these Feates · · is Sprong the Feate of Geodesie, or Land Measuring. 1664 V. Wing Art Surv. 111 Geodæsie or Land-measure. 1755 Johnson, Geodæsia [citing Harris]. 1766 B. Martin Surv. by Goniom. 6 With regard to Geodesia or Land Surveying, and all kinds of Longimetry, the natural eyesight ought to be assisted. 1853 Herschel Pop. Lect. Sc. v. §13. (1873) 189 'Geodesy' as distinct from mere mensuration and surveying. 1855 J. B. Williams (title), Practical Geodesy, comprising chain surveying and the use of surveying instruments. 1881 M. Merriman (title), Figure of the Earth: an Introduction to Geodesy."

The concept of geodetic then follows (Oxford University, 2009):

> " [as if ad. L. *geōdætic-us, a. Gr. *γεωδαιτικός, f. γῆ earth + δαίειν to divide.] …
>
> A.A adj. a.A.a Of or pertaining to geodesy. geodetic line (see quot. 1879).
>
> 1834 Nat. Philos., Astron. xiii. 253/1 (U.K.S.) Those great geodetic operations which have been undertaken to determine the figure of the earth. 1879 Thomson & Tait Nat. Phil. I. i. §132 If the shortest possible line be drawn from one point of a surface to another, its plane of curvature is everywhere perpendicular to the surface. Such a curve is called a Geodetic line. 1880 Nature XXI. 197 Geographical

> and topographical work such as had been carried on by the Coast and Geodetic Surveys and the Land Office."

What I learn is how the idea of finding the shortest path is related to studying the geometry of the earth. People have been thinking about this problem for some time and have settled on a variety of techniques to approach the problem. I follow one of these techniques and begin by introducing more modern terminology and notation.

The goal will be to find the shortest path between two points on an oblate spheroid, a path we call a ***geodesic path***. In what follows, the ***polar angle*** θ is the complement of longitude, and the slope of the cross section of the earth defines the tangent of the angle ψ, which is the complement of the ***geodetic longitude***. An oblate spheroid in polar coordinates $\{r, \quad \theta, \quad \varphi\}$ is independent of the azimuth φ, and depends on a major and minor axis length a and b respectively, with $a \geq b$:

$$\left(\frac{r\sin\theta}{a}\right)^2 + \left(\frac{r\cos\theta}{b}\right)^2 = 1 \tag{A.1}$$

In other words, the cross section of the oblate spheroid is an ellipse. The ***eccentricity*** of the ellipse is defined by the major and minor axis lengths:

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{A.2}$$

It is then not difficult to re-express the radius as a function of the polar angle θ , the eccentricity e and the major axis a:

$$r(\theta) = a\sqrt{\frac{1-e^2}{1-e^2\sin^2\theta}} \tag{A.3}$$

The geodetic longitude is determined by the slope dy/dx, which is determined by the transformation to rectangular coordinates:

$$\begin{aligned}
y &= r(\theta)\cos\theta \\
x &= r(\theta)\sin\theta \\
r'(\theta) &= \frac{dr(\theta)}{d\theta} \\
\frac{dy}{d\theta} &= r'(\theta)\cos\theta - r(\theta)\sin\theta \\
\frac{dx}{d\theta} &= r'(\theta)\sin\theta + r(\theta)\cos\theta
\end{aligned} \tag{A.4}$$

The "dash" represents the derivative with respect to the polar angle. I solve for the tangent:

$$\tan\psi = -\frac{dy}{dx} = \frac{r\sin\theta - r'\cos\theta}{r\cos\theta + r'\sin\theta} = \frac{\tan\theta - \frac{r'}{r}}{1+\frac{r'}{r}\tan\theta} \tag{A.5}$$

To evaluate this, I take the definition of the oblate spheroid Eq. (A.3), and differentiate:

$$\begin{aligned} r^2 &= a^2\frac{1-e^2}{1-e^2\sin^2\theta} \\ \frac{r'}{r} &= a^2\left(1-e^2\right)\frac{e^2\sin\theta\cos\theta}{\left(1-e^2\sin^2\theta\right)^2}\frac{1}{r^2} \\ \frac{r'}{r} &= \frac{e^2\sin\theta\cos\theta}{1-e^2\sin^2\theta} \\ \tan\psi &= \left(1-e^2\right)\tan\theta \end{aligned} \tag{A.6}$$

The relationship between the ***geodetic polar angle*** ψ and the ***polar angle*** θ is thus quite simple:

$$\tan\psi = \left(1-e^2\right)\tan\theta \tag{A.7}$$

My interest is to compute the path length between any two points on the oblate spheroid. I will write this in various forms below, including using the original spherical coordinates as well as a variety of other coordinate systems. Each provides some insight into how to compute the shortest path.

The differential distance between two points is determined from the spherical coordinate form:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 = \left(r'^2 + r^2\right)d\theta^2 + r^2\sin^2\theta d\varphi^2 \tag{A.8}$$

I make use of the constraint that the radius and polar angle are related on the spheroid. This is the most straightforward form of the line element.

To make a connection with map making and the Mercator projection, I proceed as follows. I start with some useful intermediate formulae relating these angles, and for this I introduce a different way to write the eccentricity:

$$\kappa^2 \equiv 1-\left(1-e^2\right)^2 = 2e^2 - e^4 = 1-\left(\frac{b}{a}\right)^4 \tag{A.9}$$

I then compute the sine and cosine of the polar angle and the geodetic polar angle, finding identities that will be useful in computing the differential distance $dr^2 + r^2 d\theta^2 = \left(r'^2 + r^2\right) d\theta^2$:

$$
\begin{aligned}
&\tan^2\psi = \frac{\sin^2\psi}{1-\sin^2\psi} \Rightarrow \sin^2\psi = \frac{\tan^2\psi}{1+\tan^2\psi} \\
&\frac{1}{\tan^2\psi} = \frac{1}{\sin^2\psi} - 1 \\
&\sin^2\psi = \frac{\left(1-e^2\right)^2 \tan^2\theta}{1+\left(1-e^2\right)^2 \tan^2\theta} = \frac{\left(1-\kappa^2\right)\sin^2\theta}{1-\kappa^2\sin^2\theta} \\
&\cos^2\psi = \frac{\cos^2\theta}{1-\kappa^2\sin^2\theta} \\
&\sin^2\theta = \frac{\sin^2\psi}{1-\kappa^2\cos^2\psi} \\
&\cos^2\theta = \frac{\left(1-\kappa^2\right)\cos^2\psi}{1-\kappa^2\cos^2\psi}
\end{aligned}
\tag{A.10}
$$

I determine the quantities that appear in Eq. (A.8), namely the ratio of the radial derivative to the radius and $r^2 \sin^2\theta$ as follows:

$$
\begin{aligned}
&\sin\theta\cos\theta = \frac{\left(1-e^2\right)\sin\psi\cos\psi}{1-\kappa^2\cos^2\psi} \\
&\kappa^2 - e^2 = 1 - e^2 - \left(1-e^2\right)^2 = e^2\left(1-e^2\right) \\
&1 - e^2\sin^2\theta = \frac{1-\kappa^2\cos^2\psi - e^2\sin^2\psi}{1-\kappa^2\cos^2\psi} = \left(1-e^2\right)\frac{1-e^2\cos^2\psi}{1-\kappa^2\cos^2\psi} \\
&\frac{r'}{r} = \frac{e^2\sin\theta\cos\theta}{1-e^2\sin^2\theta} = \frac{e^2\left(1-e^2\right)\sin\psi\cos\psi}{\left(1-e^2\right)\left(1-e^2\cos^2\psi\right)} = \frac{e^2\sin\psi\cos\psi}{1-e^2\cos^2\psi} \\
&r^2 = a^2\frac{1-e^2}{1-e^2\sin^2\theta} = a^2\frac{1-\kappa^2\cos^2\psi}{1-e^2\cos^2\psi} \\
&r^2\sin^2\theta = a^2\frac{1-\kappa^2\cos^2\psi}{1-e^2\cos^2\psi}\frac{\sin^2\psi}{1-\kappa^2\cos^2\psi} = a^2\frac{\sin^2\psi}{1-e^2\cos^2\psi}
\end{aligned}
\tag{A.11}
$$

I get the following contribution for the radius and longitudinal components using these relationships:

$$r^2 + r'^2 = r^2\left(1+\left(\frac{r'}{r}\right)^2\right)$$

$$r^2 + r'^2 = a^2\frac{1-\kappa^2\cos^2\psi}{\left(1-e^2\cos^2\psi\right)^3}\left(1-2e^2\cos^2\psi+e^4\cos^4\psi+e^4\sin^2\psi\cos^2\psi\right) \tag{A.12}$$

$$r^2 + r'^2 = a^2\frac{1-\kappa^2\cos^2\psi}{\left(1-e^2\cos^2\psi\right)^3}\left(1-\kappa^2\cos^2\psi\right) = a^2\frac{\left(1-\kappa^2\cos^2\psi\right)^2}{\left(1-e^2\cos^2\psi\right)^3}$$

The next piece of the puzzle is to express this relationship using the Mercator projection. The ***Mercator Projection*** M is defined in terms of the geodetic polar angle:

$$\sin\psi\frac{dM}{d\psi} \equiv -\frac{1-e^2}{1-e^2\cos^2\psi} \tag{A.13}$$

This gives for the projection M the following specific expression in terms of the ***geodetic polar angle***:

$$M = -\ln\tan\tfrac{1}{2}\psi - \tfrac{1}{2}e\ln\frac{1+e\cos\psi}{1-e\cos\psi} \tag{A.14}$$

I also have the identities:

$$\begin{aligned}
&\tan\psi = \left(1-e^2\right)\tan\theta \\
&\frac{d\psi}{\cos^2\psi} = \left(1-e^2\right)\frac{d\theta}{\cos^2\theta} \\
&\frac{d\psi}{d\theta} = \frac{1-k^2\cos^2\psi}{1-e^2}
\end{aligned} \tag{A.15}$$

With these, I compute the line element Eq. (A.8):

$$ds^2 = \frac{a^2}{1-e^2\cos^2\psi}\left(\left(\frac{1-e^2}{1-e^2\cos^2\psi}\right)^2 d\psi^2 + \sin^2\psi\, d\varphi^2\right) \tag{A.16}$$

I am close to complete; the last step is that I write this expression in terms of the Mercator projection M .

Consider the length ρ defined purely as a function of the geodetic polar angle, Figure 1-2

$$r\cos\theta \equiv \rho\cos\psi \Rightarrow \rho^2 = r^2\frac{\cos^2\theta}{\cos^2\psi} = \frac{a^2\left(1-e^2\right)^2}{1-e^2\cos^2\psi} \tag{A.17}$$

In terms of this unit of length, the differential distance between two points on an oblate spheroid is:

$$ds^2 = \frac{\rho^2}{\left(1-e^2\right)^2}\left(\left(\frac{1-e^2}{1-e^2\cos^2\psi}\right)^2 d\psi^2 + \sin^2\psi d\varphi^2\right) \tag{A.18}$$

This can be written in terms of the Mercator projection to give the ***Mercator Line Element***:

$$ds^2 = \frac{\rho^2\sin^2\psi}{1-\kappa^2}\left(dM^2 + d\varphi^2\right) \tag{A.19}$$

In this form, the Mercator projection and the longitude provide measures of distance that looks Euclidean as long as the geodetic polar angle is constant. Remember that the geodetic polar angle is determined by the Mercator M based on the differential equation Eq. (A.13).

To understand better the origin of this approach, I refer to the Oxford English Dictionary (Oxford University, 2009) on the word Mercator:

> a.”a The name of Gerhardus Mercator (= L. equivalent of Gerhard Kremer) (1512–94), Flemish cartographer, used attrib. and in the possessive with reference to the orthomorphic cylindrical map projection first used by him in 1568, in which meridians are represented by equidistant straight lines at right angles to the equator and any course that follows a constant compass bearing is represented by a straight line. …
>
> 1669 Mercator's projection [see projection n. 7 b]. a 1877 Knight Dict. Mech. II. 1419/2 Mercator-chart, a mode of projection invented by Gerald Mercator, in which the meridians and parallels are straight and parallel lines. 1883 Encycl. Brit. XV. 520/2 By 1601 Mercator's projection was in use for all sea charts. 1908 G. R. Putnam Naut. Charts 9 The Arcano del Mare, 1646, was the first marine atlas in which all the maps were drawn on the mercator projection. 1912 A. R. Hinks Map Projections iii. 29 The great distortion in the north and south makes Mercator's projection altogether unsuitable for a land map. 1938 L. M. Milne-Thomson Theoret. Hydrodynamics v. 138 An illustration of conformal mapping is afforded by an ordinary map on Mercator's projection. 1960 C. Eckart Hydrodynamics of Oceans & Atmospheres 280 The vertical co-ordinate, μ, on a Mercator chart of the Sphere is defined by dϕ = cosϕdμ.”

The value of the Mercator Line Element is that on a chart, distances are Euclidean in a small area around any point: locally, the shortest distances are straight lines. The Mercator projection M and the azimuth φ are on the same footing; they have the same metric factor.

Spherical Trigonometry

I have provided the vocabulary for describing points on an oblate spheroid. Next I need a way to compute the shortest path. I recall the rules of spherical trigonometry, which consist of two general laws: the law of sines and the law of cosines. From these two laws the behaviors of paths on a sphere are completely determined. Is there something similar for an oblate spheroid? I believe I have found such a relationship, though I don’t really have a detailed set of references

for it yet. I define a new "polar angle" f , which will play the role of a "spherical" polar angle for the oblate spheroid:

$$r\sin\theta = a\sin f \tag{A.20}$$

This is well defined for a point on the oblate spheroid as long as a is the major axis so $r/a \le 1$. The following result then follows:

$$\frac{\rho^2 \sin^2\psi}{1-\kappa^2} = a^2 \sin^2 f \tag{A.21}$$

In other words, the scale factor in front of the Mercator and azimuth terms in Eq. (A.19) depends only on the sine of this new angle. I can also show that the sines and cosines are related as follows:

$$\begin{aligned}
\sin^2 f &= \frac{\sin^2\psi}{1-e^2\cos^2\psi} \\
\cos^2 f &= \left(1-e^2\right)\frac{\cos^2\psi}{1-e^2\cos^2\psi} \\
\tan^2 f &= \frac{1}{1-e^2}\tan^2\psi = \left(1-e^2\right)\tan^2\theta
\end{aligned} \tag{A.22}$$

With such relationships, I can express the line element in terms of this new angle:

$$\begin{aligned}
\frac{df}{\cos^2 f} &= \sqrt{\frac{1}{1-e^2}}\frac{d\psi}{\cos^2\psi} \\
\left(\frac{d\psi}{df}\right)^2 &= \left(1-e^2\right)\left(\frac{\cos^2\psi}{\cos^2 f}\right)^2 = \frac{1}{\left(1-e^2\right)}\left(1-e^2\cos^2\psi\right)^2 \\
ds^2 &= a^2\frac{\left(1-e^2\right)}{\left(1-e^2\cos^2\psi\right)}df^2 + a^2\sin^2 f d\varphi^2 \\
1-e^2\sin^2 f &= 1-\frac{e^2\sin^2\psi}{1-e^2\cos^2\psi} = \frac{1-e^2}{1-e^2\cos^2\psi} \\
ds^2 &= a^2\left(1-e^2\sin^2 f\right)df^2 + a^2\sin^2 f d\varphi^2
\end{aligned} \tag{A.23}$$

I study the shortest paths for the line element:

$$ds^2 = a^2\left(1-e^2\sin^2 f\right)df^2 + a^2\sin^2 f d\varphi^2 \tag{A.24}$$

It is not as simple as the Mercator line element, but is simpler than reverting back to the azimuth and polar angles directly. My hope, based on the comments from the Oxford English Dictionary is that the shortest paths will involve the bearing.

Shortest Path Computation

I consider the problem of computing the shortest path in the standard way by looking at all paths and picking the one that minimizes the distance. In mathematical notation this is:

$$\delta\int ds = \delta\int\sqrt{g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}}d\tau = 0 \tag{A.25}$$

The "metric" elements g_{ab} are read directly from the expression of the line element in any one of the bases defined, which for illustration I take to be Eq. (A.24), where there is an implicit sum over repeated indices and the index values $\{a,b\}$ span the variables f and φ. I define the function $L\left[x^a,\dot{x}^a\right]$ as depending on the longitude and latitude variables and their rate of change along the path $\dot{x}^a \equiv dx^a/d\tau$, assuming positions along the path are parameterized with respect to a scalar number τ:

$$L = g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = g_{ab}\dot{x}^a\dot{x}^b = a^2\left(1-e^2\sin^2 f\right)\dot{f}^2 + a^2\sin^2 f\,\dot{\varphi}^2 \tag{A.26}$$

There are many textbooks devoted to finding the minimum for such a problem using the form (A.25). The steps are summarized below:

$$\begin{aligned}
&\pi_a = \frac{\delta\sqrt{L}}{\delta\dot{x}^a}, \quad \dot{\pi}_a = \frac{\partial\sqrt{L}}{\partial x^a}\\
&\pi_a = \frac{1}{\sqrt{L}}g_{ab}\dot{x}^b\\
&\dot{\pi}_a = \tfrac{1}{2}\frac{1}{\sqrt{L}}\frac{\partial L}{\partial x^a} = \tfrac{1}{2}\frac{1}{\sqrt{L}}\frac{\partial g_{bc}}{\partial x^a}\dot{x}^b\dot{x}^c\\
&-\tfrac{1}{2}\frac{\dot{L}}{L^{3/2}}g_{ab}\dot{x}^b + \frac{1}{\sqrt{L}}g_{ab}\ddot{x}^b + \frac{1}{\sqrt{L}}\dot{g}_{ab}\dot{x}^b = \tfrac{1}{2}\frac{1}{\sqrt{L}}\frac{\partial g_{bc}}{\partial x^a}\dot{x}^b\dot{x}^c\\
&-\tfrac{1}{2}\frac{\dot{L}}{L}g_{ab}\dot{x}^b + g_{ab}\ddot{x}^b + \tfrac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)\dot{x}^b\dot{x}^c = 0
\end{aligned} \tag{A.27}$$

The first line follows from varying the positions and the derivatives separately, then integrating by parts to identify only the variations of the functions:

$$
\begin{aligned}
\delta\int\sqrt{L}d\tau &= \int\left(\frac{\delta\sqrt{L}}{\delta x^a}\delta x^a + \frac{\delta\sqrt{L}}{\delta \dot{x}^a}\delta \dot{x}^a\right)d\tau \\
\delta\int\sqrt{L}d\tau &= \int\left(\frac{\delta\sqrt{L}}{\delta x^a}\delta x^a - \frac{d}{d\tau}\frac{\delta\sqrt{L}}{\delta \dot{x}^a}\delta x^a\right)d\tau \\
\delta\int\sqrt{L}d\tau &= \int\left(\frac{\delta\sqrt{L}}{\delta x^a} - \frac{d}{d\tau}\frac{\delta\sqrt{L}}{\delta \dot{x}^a}\right)\delta x^a d\tau \\
\frac{\delta\sqrt{L}}{\delta x^a} - \frac{d}{d\tau}\frac{\delta\sqrt{L}}{\delta \dot{x}^a} &= 0
\end{aligned}
\tag{A.28}
$$

The second line of Eq. (A.27) defines the "momentum" variable π_a , whose rate of change is determined by the variation equation, and evaluates the various derivatives that occur. Note in the end, there is the leading term proportional to $\dot{L}$ that depends on the variation of the path length as you move along the path based on the parameter τ . There is no loss in generality to take that parameter to be the path length, in which case the resultant equations hve $\dot{L}=0$ and are called the ***geodesic equations***:

$$
g_{ab}\ddot{x}^b + \tfrac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)\dot{x}^b\dot{x}^c = 0 \tag{A.29}
$$

These results generalize to arbitrary spaces with arbitrary metrics, though we have derived it for the simpler case of an oblate spheroid.

Since there are two coordinates, there are two equations. We have one such equation for f and one such equation for φ . The simplest equation is that for the azimuth φ since in that case the metric components are independent of φ , so the momentum along that direction is conserved:

$$
\begin{aligned}
\dot{\pi}_\varphi &= \tfrac{1}{2}\frac{1}{\sqrt{L}}\frac{\partial g_{bc}}{\partial\varphi}\dot{x}^b\dot{x}^c = 0 \\
\pi_\varphi &= a^2\sin^2 f\,\dot{\varphi} = a^2\sin^2 f\,\frac{d\varphi}{ds} = const.
\end{aligned}
\tag{A.30}
$$

I have also assuming that L is unity everywhere along the path length. I use this fact with the form for the Mercator line element Eq. (A.19) and with the common factor given by (A.21):

$$
L = \left(a\frac{dM}{ds}\sin f\right)^2 + \left(a\frac{d\varphi}{ds}\sin f\right)^2 = 1 \tag{A.31}
$$

I solve this in terms of an angle α:

$$a\frac{dM}{ds}\sin f = -\cos\alpha$$
$$a\frac{d\varphi}{ds}\sin f = \sin\alpha \tag{A.32}$$

The interpretation is that α is the ***bearing*** or angle along the geodesic. In terms of the bearing, the conservation law Eq. (A.30) is the ***law of sines***:

$$\sin f_0 \sin\alpha_0 = \sin f \sin\alpha \tag{A.33}$$

This generalizes the law of sines from spherical trigonometry. It provides an interpretation of the angle f consistent with the spherical trigonometry picture Figure 1-3.

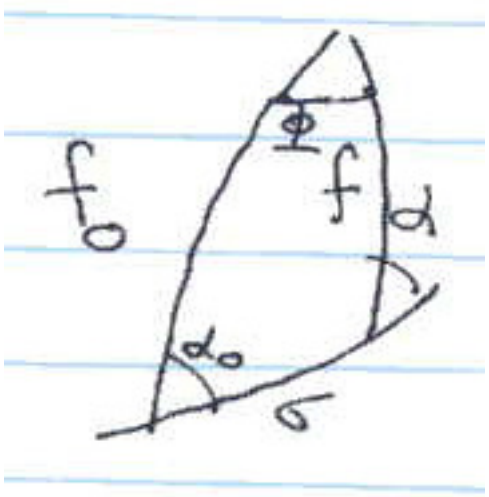

Figure 1-3: Quasi-spherical triangle

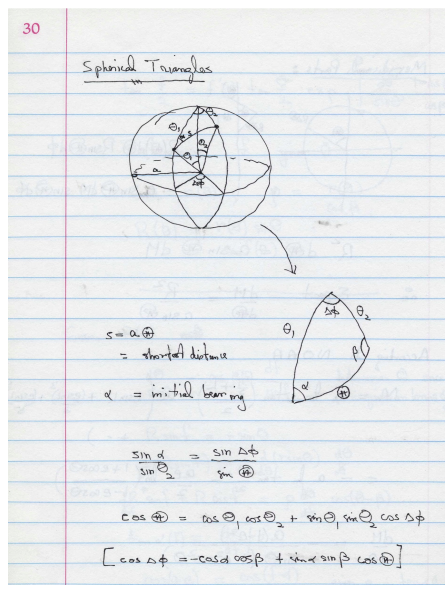

Figure 1-4: Spherical Triangle

I derive the relationship between bearing, Meridian and azimuth from Eq. (A.32):

$$\frac{d\varphi}{dM} = -\tan\alpha \tag{A.34}$$

This says that as I move along a geodesic with constant bearing on a Mercator chart, it is a straight line. Of course I have not shown that geodesics lead to constant bearings. In fact I will show that this is not true.

Another interesting relationship is the variation of the bearing with path length along a geodesic:

$$
\begin{aligned}
&\sin\alpha\sin f = const.\\
&\cos\alpha\sin f\,\frac{d\alpha}{ds} = -\sin\alpha\cos f\,\frac{df}{ds}\\
&\frac{df}{\cos^2 f} = \sqrt{\frac{1}{1-e^2}}\,\frac{d\psi}{\cos^2\psi}\\
&\sin\psi\,\frac{dM}{d\psi} = -\frac{1-e^2}{1-e^2\cos^2\psi}\\
&-a\frac{dM}{ds}\sin f\sin f\,\frac{d\alpha}{ds} = -a\frac{d\varphi}{ds}\sin f\cos f\,\frac{df}{ds}\\
&\sin f\,\frac{d\alpha}{ds} = \frac{d\varphi}{ds}\cos f\,\frac{df}{dM} = \frac{d\varphi}{ds}\cos f\,\frac{df}{d\psi}\frac{d\psi}{dM}\\
&\sin f\,\frac{d\alpha}{ds} = -\frac{d\varphi}{ds}\cos f\sqrt{\frac{1}{1-e^2}}\,\frac{\cos^2 f}{\cos^2\psi}\frac{\sin\psi}{1-e^2}\left(1-e^2\cos^2\psi\right)\\
&\tan f\,\frac{d\alpha}{ds} = -\sin\psi\,\frac{d\varphi}{ds}\sqrt{\frac{1}{1-e^2}} = \sqrt{\frac{1}{1-e^2}}\tan\psi\,\frac{d\alpha}{ds}\\
&\frac{d\alpha}{ds} = -\frac{d\varphi}{ds}\cos\psi
\end{aligned}
\tag{A.35}
$$

It suggests that if when the bearing doesn't change along the geodesic, we are going along a path of fixed azimuth; in other words a geodetic longitude.

Quasi-spherical geometry

I still need to provide the equations of motion for the bearing and at the same time provide some insight. My goal is to generalize the spherical trigonometry relationship called the law of cosines which, with the initial conditions, provides the geodesic paths. I start with the line element Eq. (A.24). The two equations of motion are Eq. (A.32):

$$
\begin{aligned}
&a\sqrt{1-e^2\sin^2 f}\,\frac{df}{ds} = \cos\alpha\\
&a\sin f\,\frac{d\varphi}{ds} = \sin\alpha
\end{aligned}
\tag{A.36}
$$

I introduce constants $\{\beta,\gamma\}$ which define initial conditions as well as a new angular path variable σ:

$$
\begin{aligned}
&\sin\alpha\sin f = \sin\alpha_0\sin f_0 \equiv \sqrt{1-\gamma^2}\\
&\gamma\cos\beta \equiv \cos f_0\\
&\gamma\cos(\beta+\sigma) \equiv \cos f
\end{aligned}
\tag{A.37}
$$

As f varies, I see that σ varies.

The meaning of the parameter γ is that it measures the conserved angular momentum around the symmetry axis:

$$\sin^2 f \frac{d\varphi}{ds} = \sqrt{1-\gamma^2} \tag{A.38}$$

The smaller the angular momentum, (i.e. the closer γ is to unity), the slower the change in the azimuth, and hence the faster the geodesic will climb northward compared to going east or west. Conversely, the higher the angular momentum, the easterly or westerly will be the motion. The bound on the angular momentum is that within the integrands above, $\gamma_0^2 \geq \cos^2 f$. Equality as an option would have to be studied to see if it makes sense.

I differentiate the defining relationship for σ along the geodesic:

$$\begin{aligned} &\gamma \cos(\beta+\sigma) = \cos f \\ &-\gamma \sin(\beta+\sigma)\frac{d\sigma}{df} = -\sin f \\ &\sin f \frac{df}{d\sigma} = \gamma \sin(\beta+\sigma) \end{aligned} \tag{A.39}$$

I convert this with the defining relationships:

$$\begin{aligned} &\sin f \frac{df}{d\sigma} = \sqrt{\gamma^2 - \gamma^2 \cos^2(\beta+\sigma)} = -\sqrt{\gamma^2 - \cos^2 f} \\ &\sin f \frac{df}{d\sigma} = \sqrt{\sin^2 f - \sin^2 \alpha \sin^2 f} \\ &\sin f \frac{df}{d\sigma} = \sin f \cos \alpha \\ &\frac{df}{d\sigma} = \cos \alpha \end{aligned} \tag{A.40}$$

Next, I couple this with the other equation of motion:

$$\begin{aligned} &a\sqrt{1-e^2 \sin^2 f}\,\frac{df}{ds} = \cos\alpha = \frac{df}{d\sigma} \\ &\frac{ds}{d\sigma} = a\sqrt{1-e^2 \sin^2 f} = a\sqrt{1-e^2+e^2\gamma^2\cos^2(\beta+\sigma)} \end{aligned} \tag{A.41}$$

In other words, the angle σ is the ***angular path length***. The remaining equation of motion is:

$$\begin{aligned}
&\sin f \frac{d\varphi}{d\sigma} = \sin\alpha\sqrt{1-e^2\sin^2 f} \\
&\frac{d\varphi}{d\sigma} = \frac{\sqrt{1-\gamma^2}\sqrt{1-e^2\sin^2 f}}{\sin^2 f} = \sqrt{1-\gamma^2}\frac{\sqrt{1-e^2+e^2\gamma^2\cos^2(\beta+\sigma)}}{1-\gamma^2\cos^2(\beta+\sigma)}
\end{aligned} \tag{A.42}$$

Thus, based on initial conditions I compute the azimuth and the actual path length. The defining relationship Eq. (A.37) gives us f as a function of the angular path length and also by the same relationships, the bearing α. I have a complete set of equations along the geodesic.

To gain more insight I elaborate on Figure 1-3 and introduce a new azimuth variable Φ :

$$\sin f \frac{d\Phi}{d\sigma} \equiv \sin\alpha \tag{A.43}$$

I write out the line element using this variable:

$$\begin{aligned}
&ds^2 = a^2\left(1-e^2\sin^2 f\right)d\sigma^2 \\
&a^2\left(1-e^2\sin^2 f\right)d\sigma^2 = a^2\left(1-e^2\sin^2 f\right)df^2 + a^2\sin^2 f\left(1-e^2\sin^2 f\right)d\Phi^2 \\
&d\sigma^2 = df^2 + \sin^2 f d\Phi^2
\end{aligned} \tag{A.44}$$

The result is that the differential changes along the angular path length are related to the angles f and Φ exactly in the same way as the path on a sphere is related to the polar angle and the azimuth.

For a sphere, the geodesic path is determined by using the laws of sines and cosines based on Figure 1-3:

$$\begin{aligned}
&\frac{\sin\sigma}{\sin\Phi} = \frac{\sin f}{\sin\alpha_0} = \frac{\sin f_0}{\sin\alpha} \\
&\cos f = \cos f_0\cos\sigma + \sin f_0\sin\sigma\cos\alpha_0 \\
&\cos\sigma = \cos f_0\cos f + \sin f_0\sin f\cos\Phi
\end{aligned} \tag{A.45}$$

In spherical coordinates, if I know the initial polar angle f_0 and latitude $\Phi_0 = 0$, and the final polar angle f and latitude Φ, then I know from the law of cosines the distance σ. Based on the known distance, the law of Sines provides the initial bearing. These two pieces of information are the key to navigation. I know which way to go, and I know how far I will go. What is the corresponding procedure for an oblate spheroid? The problem is that I am given the final azimuth φ, not Φ.

I use Eqs. (A.41) and (A.42) along with the defining relationships to obtain the variation of the azimuth along the geodesic as a function of f :

$$\begin{aligned}
&\frac{d\varphi}{df}=\frac{d\varphi}{ds}\frac{ds}{df}\\
&\frac{d\varphi}{df}=\frac{\sin\alpha}{\sin f}\frac{\sqrt{1-e^2\sin^2 f}}{\cos\alpha}=\frac{\sin\alpha\sin f}{\sin^2 f}\frac{\sqrt{1-e^2\sin^2 f}}{\cos\alpha}=\frac{\sqrt{1-\gamma^2}}{\sin^2 f}\frac{\sqrt{1-e^2\sin^2 f}}{\cos\alpha}\\
&1-\sin^2\alpha\sin^2 f=1-\sin^2 f+\cos^2\alpha\sin^2 f=\gamma^2\\
&\cos\alpha=\frac{\sqrt{\gamma^2-\cos^2 f}}{\sin f}\\
&\frac{d\varphi}{df}=\frac{\sqrt{1-\gamma^2}}{\sin f}\frac{\sqrt{1-e^2\sin^2 f}}{\sqrt{\gamma^2-\cos^2 f}}\\
&d\varphi=\sqrt{1-\gamma^2}\sqrt{\frac{1-e^2\sin^2 f}{\gamma^2-\cos^2 f}}\frac{df}{\sin f}
\end{aligned}\tag{A.46}$$

Integrating this I obtain:

$$\varphi(f)=\varphi_0+\sqrt{1-\gamma^2}\int_{f_0}^{f}\sqrt{\frac{1-e^2\sin^2 f}{\gamma^2-\cos^2 f}}\frac{df}{\sin f}\tag{A.47}$$

Assuming the initial and final longitude values are known and fixed, this gives the azimuth φ as a function of the constant parameter γ. I would need to plot this, and pick the value of γ_0 that corresponds to the final azimuth. Once this is set, I know the initial bearing α_0 from Eq. (A.37). I then integrate the longitude expression Eq. (A.36) to get the total distance:

$$\begin{aligned}
&\frac{ds}{df}=a\sqrt{\frac{1-e^2\sin^2 f}{\gamma^2-\cos^2 f}}\sin f\\
&s(f)=\int_{f_0}^{f}\sin f\sqrt{\frac{1-e^2\sin^2 f}{\gamma_0^{\,2}-\cos^2 f}}df
\end{aligned}\tag{A.48}$$

This technique gives a method that can be used with Mathematica® to determine the path length and the azimuth for a given start point and a given end point. I can carry this out in any convenient frame of reference, including the original spherical reference frame Eq. (A.8).

Summary

I have discussed in some details the approaches one might take to obtaining the shortest distance between two points on a two dimensional surface: specifically an oblate spheroid. To summarize, I start with the line element

$$ds^2 = g_{ab} dx^a dx^b \tag{A.49}$$

For the oblate spheroid in the spherical basis, the metric elements are diagonal and independent of the azimuth:

$$\left\{ g_{11} \equiv g_{\theta\theta} = r^2 + r'^2 \quad g_{22} \equiv g_{\varphi\varphi} = r^2 \sin^2\theta \right\} \tag{A.50}$$

The shortest path is determined by two conservation laws, which lead to first order equations:

$$\begin{aligned} g_{\varphi\varphi} \frac{d\varphi}{d\tau} &= const \\ \sqrt{g_{\theta\theta}} \frac{d\theta}{d\tau} &= \cos\bar{\beta} \\ \sqrt{g_{\varphi\varphi}} \frac{d\varphi}{d\tau} &= \sin\bar{\beta} \end{aligned} \tag{A.51}$$

These equations can be rewritten as:

$$\begin{aligned} \sqrt{g_{\varphi\varphi}(\theta)} \sin\bar{\beta}(\theta) &= \sqrt{g_{\varphi\varphi}(\theta_0)} \sin\bar{\beta}_0 \\ d\varphi &= \sqrt{\frac{g_{\theta\theta}(\theta)}{g_{\varphi\varphi}(\theta)}} \tan\bar{\beta}(\theta) d\theta \\ d\tau &= \frac{d\tau}{d\varphi}\frac{d\varphi}{d\theta} d\theta = \frac{\sqrt{g_{\varphi\varphi}(\theta) g_{\theta\theta}(\theta)}}{\sqrt{g_{\varphi\varphi}(\theta_0)} \sin\bar{\beta}_0} \tan\bar{\beta}(\theta) d\theta \end{aligned} \tag{A.52}$$

The first equation determines the bearing $\bar{\beta}$ in terms of the polar angle, given initial conditions. The second equation can be integrated to give the azimuth at any polar angle. The last equation can be integrated to give the line element at any polar angle along the geodesic. These three equations thus summarize what we have to do for any line element whose metric is independent of azimuth.

References

Oxford University. (2009). Oxford English Dictionary Second Edition on CD ROM (v4.0). *OED* . Oxford, England: Oxford University Press.

Thomas, G. H. (2006). *Geometry, Language and Strategy.* New Jersey: World Scientific.