Decision Making, Chaos and Determinism

Gerald H. Thomas

Milwaukee School of Engineering

June, 2013

B-1. Introduction

Decision making is one of the most human acts and seems to be the most difficult area to formalize into a theory of behaviors that are causal and deterministic. In fact one might think that the very nature of decision making is one of chance and uncertainty. One issue we think relevant is the general lack of understanding of causal theories and how they deal with uncertainty. Moreover, in our view, there is insufficient appreciation of the sensitivity of the initial conditions that determine future behaviors. When these issues are taken into account, it becomes easier to see the possibility for causal formal theories of human decision making.

Consider the sensitivity of future behaviors on initial conditions, which has been extensively studied under the general category of ***chaos*** and chaos theory. It has been said in the past that chaos represents for humans the way we perceive the world in its unordered state. A recent definition, according to the OED (Oxford University, 2009), is:

> Add: [3.] c.3.c Math. Behaviour of a system which is governed by deterministic laws but is so unpredictable as to appear random, owing to its extreme sensitivity to changes in parameters or its dependence on a large number of independent variables; a state characterized by such behaviour.

If we had perfect information, so the argument goes, we would have perfect determinism. The quote on chaos makes this argument seem unattainable. Slight disturbances on what we think we know, lead to unknown consequences, even in a theory that is strictly causal and deterministic. So "what is deterministic?" It seems to us to be unreasonable to believe that because chaos behavior is possible, we must throw out our causal theories. They work very well and explain a host of data. We believe that a more reasonable approach is that we must be more careful about what we claim to learn from causal theories. The theories after all reflect our efforts to identify concepts and attributes that don't change with time or that change with time in an understandable, causal and continuous way.

One thing that might help our understanding of this issue is to explore how we deal with uncertainty in such mundane activities as measurements, which form the basis of physical theories. So for example we understand by length, an attribute that characterizes the height, width or depth of something. It is a great accomplishment in understanding to separate this concept from the mechanisms by which we perform the measurement. The mechanisms involve a measuring stick and us as agent; for those of you measuring a basement, you know that multiple measurements yield multiple answers. Yet we are confident that the basement has well defined dimensions. How did we come to this conclusion and how did we learn to separate out the uncertainties associated with us as agent and the intermediary of a measurement stick, from the invariant concept of length? I think we all agree that the separation has been done and we are comfortable with the idea of length.

B-2. Pendulum

Similarly, we are comfortable with the concept of time, despite our dependence of using clocks to make time measurements. From such simplistic considerations, we have adopted over many centuries, physical theories of the behavior of matter that we depend on. For example, we are comfortable with a host of physics problems that relate distances objects travel with time. We believe we understand how a pendulum works because we can predict the behavior starting from a description in which we describe its restoring force $\sin q$ as being the source of the acceleration $\ddot{q}$, Figure 1. The behavior is the set of position of the pendulum over time. We could start the pendulum at rest and "drive" it by a harmonic force. We then predict from Newton's theory where the pendulum will be at any future moment. We compare where the pendulum is by measurements against where it is predicted to be and find agreement to a high degree of accuracy. So we say we understand the behavior.

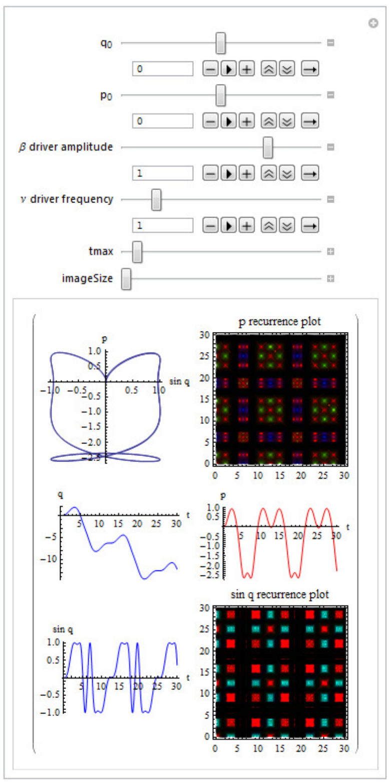

Figure 1: Pendulum

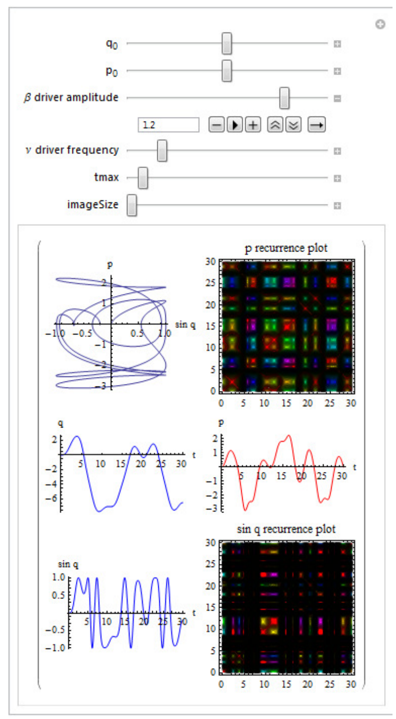

Figure 2: Pendulum with start of chaotic behaviors

There are situations however when our predictions are not as expected: if we "drive" the pendulum too hard, it demonstrates chaotic behaviors (Figure 2). We see some evidence for this in these figures. Indeed, the figures (produced from the associated CDF file) demonstrate several additional properties.

- Despite the simplicity of the problem, there are many approximations that are often made which hide the chaotic character. For example, one often assumes that the variable q, which represents the angle of the pendulum, is small and so the restoring force $\sin q$ is approximated by the angle q itself. The resulting equations are then linear. They don't exhibit chaotic behaviors.
- Although in its complete form the equations are not linear, the force is still periodic. The periodic nature manifests itself as a behavior in which the pendulum goes through complete turns.
- The periodic behavior is hidden by looking at the phase space plot of the velocity $\dot{q}$ versus the force $\sin q$. The force by definition is the rate at which the momentum ($p = \dot{q}$) changes, $\dot{p}$.

- These various attributes are shown by the behavior with time of the angle, q, the momentum p and the force $\sin q$.
- The causal nature of the problem can also illustrated by looking at recurrence plots, which are contours. In this case the contours are for the momentum difference $\left|p(t)-p(t')\right|$ at points of time t and t'. Random behavior would show no regular structure; the causal structure of the time series produces a regular pattern.

This model, because of the non-linear behavior of the force, generates a structured phase space pattern. In engineering and business, there are distinct ways to gain access to a system's non-linear characteristics. For the pendulum, one can initiate the non-linear behavior by varying the initial conditions. Alternatively, one can start the system in equilibrium and "drive" the behavior by applying an external force. In the above example, we have provided an external force $\beta \sin ft$ characterized by an amplitude β and a frequency f. As we vary the frequency and amplitude we stress the non-linear structures of the problem. For sufficiently large amplitudes we generate chaotic structures: we go from a quasi-periodic structure to one that no longer appears periodic. We see behaviors that appear much more erratic and lack the periodic behaviors seen with smaller driver amplitudes. The idea is that these properties may in fact carry over into the realm of decision making.

B-3. Game Theory, Decision Process Theory and Uncertainty

We expect that decision making has attributes that involve imperfect information as well as perfect information. The challenge is to identify each of these, separating out those attributes that have a predictable behavior from those attributes that are inherently uncertain. In this matter, we closely follow the tradition of physical theories. We have adopted ***game theory*** (Von Neumann & Morgenstern, 1944) in which an ***intrinsic view*** of decisions is a productive starting point where, we separate out the pure strategies as things of permanent interest. A pure strategy is the complete set of moves one would carry out in a decision process taking into account the moves of all of the other players or agents in the process along with any physical or chance effects that might occur. It is a complete accounting of what you would do, a complete plan given every conceivable condition. It is furthermore assumed that in practice, you can approximate this complete list with a relatively small list of pure strategies.

Just because there are pure strategies, there is no reason to believe that one of these pure strategies is the right choice to make. If you are in a competitive situation, there may be a downside to your competitor knowing that you will pick one of these strategies. The solution is to "hide" your choice by picking the pure strategies with a specific frequency. Game theory determines for you what these frequencies are without informing your opponent which choice you actually will make on any given play. The frequencies are of permanent interest and the actual choice remains uncertain.

Thus your decision choice is a specific frequency choice and in that sense represents the measurement of "length" despite the fact that in a real decision process, like a real measurement process, there are lots of uncertainties. You would like to determine the frequency choices the other players make and they want to understand your choices. We emphasize that knowing these frequency choices is not the same thing as knowing what you will actually do on a given play. We take this knowledge in the same way we take the knowledge about the size of our basement. We know how to get a good approximate set of measurements. We know that out basement has a size. For each measurement process we don't know what size we will actually get.

Game theory is static; it states what the equilibrium frequencies will be. We extend *game theory* to ***decision process theory*** (Thomas G. H., Geometry, Language and Strategy, 2006) in which the strategy frequencies vary with time. This theory predicts future frequency values based on a given set of initial conditions. The theory is causal for the time evolution of the frequencies, without actually dictating what a player will actually do at any given moment. The basis of the theory has some similarities to physics and even more underlying similarities to mathematical models of physical processes. Just as in physics, there can be external forces that dictate how the rates of change of frequencies change in time. There will be stationary situations in which these rates of change don't change, in which the forces generating such changes are zero. We equate the static scenario with the literature of *game theory* and its consideration of ***static games***: the frequencies are fixed numbers. *Static games* provide an important foundation for our approach, though our results diverge once dynamic effects are included.

A second scenario is one in which the fields that generate the forces are static, but the flows, the rates of change of the frequencies, are dynamic. The flows may depend on what other players are doing, and so we can distinguish a special subset of flows that are stationary: at a specific "location", the flow doesn't change. However, if you follow the streamline of the flow you will follow a path that changes in time. You might think of a weather pattern that is stationary, in which the wind at any position is constant in speed and direction. If you follow a path along the wind by adding smoke, you will see that the smoke follows a streamline that moves with time. How would this scenario play out compared to our example with the pendulum problem?

Mathematical details

We use a mathematical vocabulary to articulate the model (Thomas G. H., 2013). This section is reference so that you don't have to dig for the details on a first pass.

Suppose we represent the rates of change of frequencies by the flow vector V^a. The upper "index" a represents a pure strategy that can be chosen by one of the players. Each pure strategy is a decision that is made by one of the participants to the decision process, the players. Subsets of these decisions are thus "owned" by specific players or participants. The acceleration of these flows is then determined (Thomas G. H., 2013):

$$g_{ab}\frac{dV^b}{d\tau}+\omega_{abc}V^bV^c=V_kF^k{}_{ab}V^b+\tfrac{1}{2}V^jV^k\partial_a\gamma_{jk} \tag{B.1}$$

The notation is that we sum over repeated indices (whenever one is upper and one is lower) of the pure strategies and that the player indices are labeled $j,k,...$, whereas the active strategies, including time, are labeled $a,b,c...$. We generalize the concept of pure strategies to reflect "space" so that the active strategies span a space of the same dimension as the number of pure strategies for the entire set of participants. There is also a corresponding inactive space corresponding to the players. Since players can form groups, this effectively makes that space one of potential mixtures or linear combinations of pure strategies.

Finally, in a static field approximation, time itself acts as a "hedge" player labeled by the index 0, though it owns no strategies and makes no decisions. This means we can start with Eq. (B.1) with one additional player, the "hedge". For our models, it will be convenient to have a "driver" that is time dependent however. The symmetry is thus broken. To take this into account, we allow the component g_{00} to have a time-dependence:

$$a,b,c,d\neq 0$$

$$\frac{dV^a}{d\tau}=\begin{pmatrix}V_kg^{ab}\partial_bA_0^k\bar{V}^0+g^{ab}V_k\bar{F}^k_{bc}V^c+g^{ab}\bar{V}_0\bar{F}^0_{bc}V^c+\tfrac{1}{2}V^jV^kg^{ab}\partial_b\gamma_{jk}\\ +\tfrac{1}{2}g^{ab}\bar{V}^0\bar{V}^0\partial_bg_{00}-\tfrac{1}{2}g^{ab}\left(-\partial_b\bar{g}_{cd}+\partial_c\bar{g}_{db}+\partial_d\bar{g}_{bc}\right)V^cV^d\\ -\tfrac{1}{2}g^{ab}\bar{A}^0_bV^0V^0\partial_0g_{00}\end{pmatrix} \tag{B.2}$$

$$\frac{d\bar{V}_0}{d\tau}=\tfrac{1}{2}\left(1-\bar{g}_{ab}V^aV^b-g_{00}\bar{A}^0_a\bar{A}^0_bV^aV^b\right)\partial_0\ln g_{00}$$

The rate equations split into two sets; one for the pure strategies and one for time. If the time dependence is absent, then the second equation is a conservation law, which we identify with "energy". We put "bars" over the new fields to distinguish them from the old, and note these fields are related to each other. For reference, we write the relationships:

$$
\begin{aligned}
&a,b \neq 0 \\
&\overline{g}_{ab} = g_{ab} - g_{00}\overline{A}_a^0\overline{A}_b^0 \Rightarrow \overline{g}^{ab} g_{bc} = \delta_c^a \\
&\overline{g}_{00} = g_{00} \\
&g^{ab} = \overline{g}^{ab} \\
&g^{00} = \overline{g}_{00}{}^{-1} + \overline{g}^{ab}\overline{A}_a^0\overline{A}_b^0 \\
&A_a^k = \overline{A}_a^k + A_0^k\overline{A}_a^0 \\
&F_{ab}^k = \partial_a A_b^k - \partial_b A_a^k = \overline{F}_{ab}^k + A_0^k\overline{F}_{ab}^0 + \overline{A}_b^0\partial_a A_0^k - \overline{A}_a^0\partial_b A_0^k \\
&F_{a0}^k = \partial_a A_0^k \\
&\overline{V}^0 = V^0 + \overline{A}_a^0 V^a \\
&\overline{V}_0 = g_{00}\overline{V}^0 + V_k A_0^k \Rightarrow V^0 = g_{00}{}^{-1}\left(\overline{V}_0 - V_a A_0^a\right) - \overline{A}_a^0 V^a
\end{aligned}
\tag{B.3}
$$

With these details, the interested reader can derive these results from (Thomas G. H., 2013).

To study the potential origins of chaos in this theory, we make simplifications that retain some of the key non-linear aspects while simplifying the form. In the general theory, each of the players, including the "hedge" for time, can exert cooperative forces. Here, we include only those cooperative forces associated with time and assume that the matrix $\overline{g}^{ab}$ is constant (it will be minus the unit matrix):

$$
\begin{aligned}
&a,b \neq 0 \\
&\frac{dV^a}{d\tau} = g^{ab}\left(V_k\overline{F}_{bc}^k V^c + \overline{V}_0\overline{F}_{bc}^0 V^c + V_k\partial_b A_0^k\overline{V}^0 + e^{2\nu}\overline{V}^0\overline{V}^0\partial_b\nu - \overline{A}_b^0 e^{2\nu}V^0V^0\partial_0\nu\right) \\
&\frac{d\overline{V}_0}{d\tau} = e^{2\nu}\left(\overline{V}^0\overline{V}^0 - \overline{A}_a^0\overline{A}_b^0 V^a V^b\right)\partial_0\nu \\
&\frac{dy^a}{d\tau} = V^a \\
&\frac{dt}{d\tau} = V^0 = e^{-2\nu}\left(\overline{V}_0 - V_a A_0^a\right) - \overline{A}_a^0 V^a
\end{aligned}
\tag{B.4}
$$

We have a complete coupled set of ordinary differential equations for the variables $\left\{t, \quad y^a \quad \overline{V}_0 \quad V^a\right\}$, which can be solved using Mathematica. We describe the various effects that are included in this model.

The Model Description

The equations set the "force", with a general notion of acceleration determined as the rate of change of the flows of frequencies, to be equal to the player payoffs $\overline{F}_{ab}^k$. In other words, the force is determined by the payoffs each player k sees, weighted by its internal interest $V_k = e_k$, which we take to be constants. In this approximation, the composite payoff is the weighted sum of the payoffs by their interest, including a sum over the "hedge" player's payoff, which we term the ***characteristic payoff***. In game theory and here, there is also an effect due to the "value" of the game. Only the actual players see this value, which is expressed as a ***value payoff*** with a time component, $\partial_a A_0^k$. Each player payoff is generated from a vector potential, so for example $\overline{F}_{ab}^0 = \partial_a\overline{A}_b^0 - \partial_b\overline{A}_a^0$. For static fields, the force equations have no explicit dependence on the potentials (gauge invariance); we see explicit dependence on the potentials above only because our scalar field ν is not static. We turn to its interpretation now.

Non-linear effects are generated by this scalar function ν, which has both a space and time variation. In the theory, its origin if from the time component of the metric, $g_{00} = e^{2\nu}$; based on the metric choices, there will be an invariant path, that defines a path length τ:

$$
\begin{aligned}
& a, b \neq 0 \\
& g_{ab} \frac{dy^a}{d\tau} \frac{dy^b}{d\tau} + 2 g_{00} \bar{A}_a^0 \frac{dy^a}{d\tau} \frac{dt}{d\tau} + g_{00} \frac{dt}{d\tau} \frac{dt}{d\tau} = 1 - \gamma_{jk} V^j V^k \\
& g_{ab} = \bar{g}_{ab} + e^{2\nu} \bar{A}_a^0 \bar{A}_b^0 = -\delta_{ab} + e^{2\nu} \bar{A}_a^0 \bar{A}_b^0 \\
& g^{ab} = -\delta^{ab} \\
& \delta^{ab} = \delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}
\end{aligned}
\tag{B.5}
$$

In general the invariant length of the flow vector is constant along the path. Note that we take the strategy components to have a simple metric. The attracting field can change the notion of the time interval.

To get an idea of how this works, we pick an example game we have studied before (Thomas G. H., 2006), (Thomas & Kane, 2008) and (Thomas G. H., 2013): the prisoner's dilemma, a decision process involving two prisoners, each which can choose to confess or not confess. The payoffs are set to their values for the prisoner's dilemma, but the supplied Mathematica model can vary any of these values. None of the conclusions should depend on their specific values, however. We have dealt with the details of similar calculations elsewhere. We focus here on some of the main results and how these results relate to the chaos and determinism questions.

Prisoner's Dilemma Model and Chaos

We write the attracting field as two terms, the first being the "conservative" but non-linear attraction and the second being the time dependent "driver" term depending on an amplitude and a frequency:

$$
\nu(u) = \tfrac{1}{2} \gamma \delta^2 \ln\left(u^2 + \delta^2\right) + \beta \sin \alpha t \tag{B.6}
$$

We start with no "hedge" player. We explore the effects that result when the strategy dependency is a function only of the difference in preferences: $u = \tfrac{1}{2}\left(N_2 + C_2 - N_1 - C_1\right)$. If the two players play with the same enthusiasm, we see no difference. The attracting force comes into play if the players have a different enthusiasm in their play. The restoring force will be similar to the pendulum and so linear for small differences and non-linear effects at large distances; in this case there is no reason to suppose the forces are periodic so the analogy fails in this regard.

The simplest case is again one in which we see quasi-periodic behaviors. In Figure 3, the *value payoff* components $\partial_a A_0^k$ are provided compactly in the fifth column and their negative in the fifth row. The first four columns and rows correspond to the pure strategies of confess and not-confess for each player.

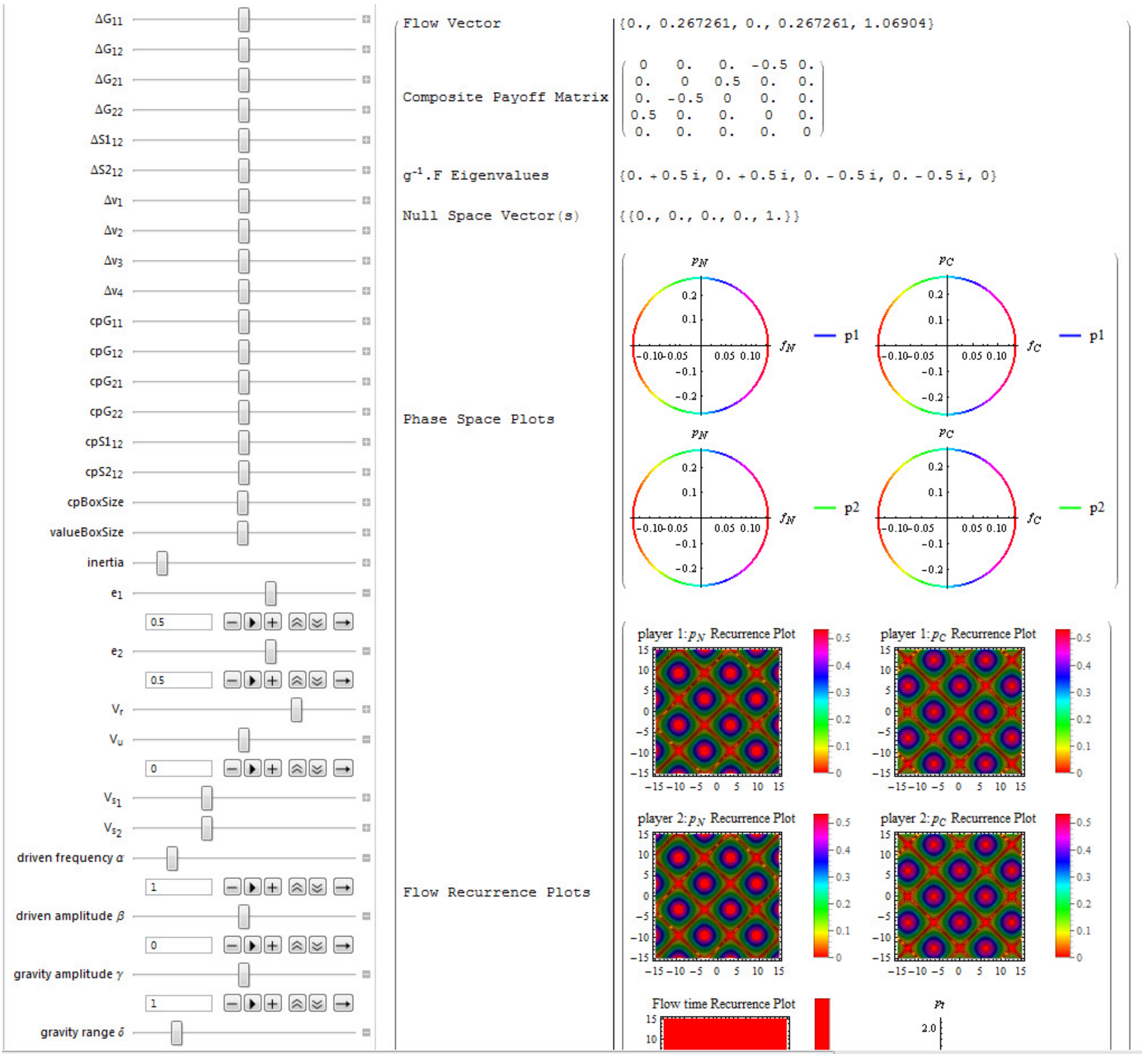

Figure 3: Baseline prisoner's dilemma

There are several items of interest to note, since there are many effects at play.

- Like any physical system or any business system, the behavior looks uninteresting only because structure effects are in near cancellation.
- The composite payoff is totally fair; the *value payoff* components $\partial_a A_0^k$ are zero. Furthermore, the game matrix, which is the sub matrix is off-diagonal with values $\pm 1/2$.
- The null space vector is the vector that generates stationary behavior; the null space is purely along the fifth direction, which we label as time.
- The phase space plots in the component subspaces are circular, indicating the harmonic behavior of the solution. This is despite the non-linear effects that are possible, but not present since we started the solution with zero flow along u, the "enthusiasm" of player 2 over player 1. In this case, we have a result similar to the pendulum above.
- The recurrence plots show the causal nature of the theory, showing a nicely periodic structure.
- For this baseline case, there is a slight tendency for each player to not-confess, opposite to the results of game theory for the prisoner's dilemma. This result is sensitive to the parameters, as you can see by playing with the model.

There are two ways to access the non-linear effects that are hiding inside this very calm structure: change the player enthusiasm flow or make the time dependent amplitude β non-zero. Here is the result of making the enthusiasm non-zero.

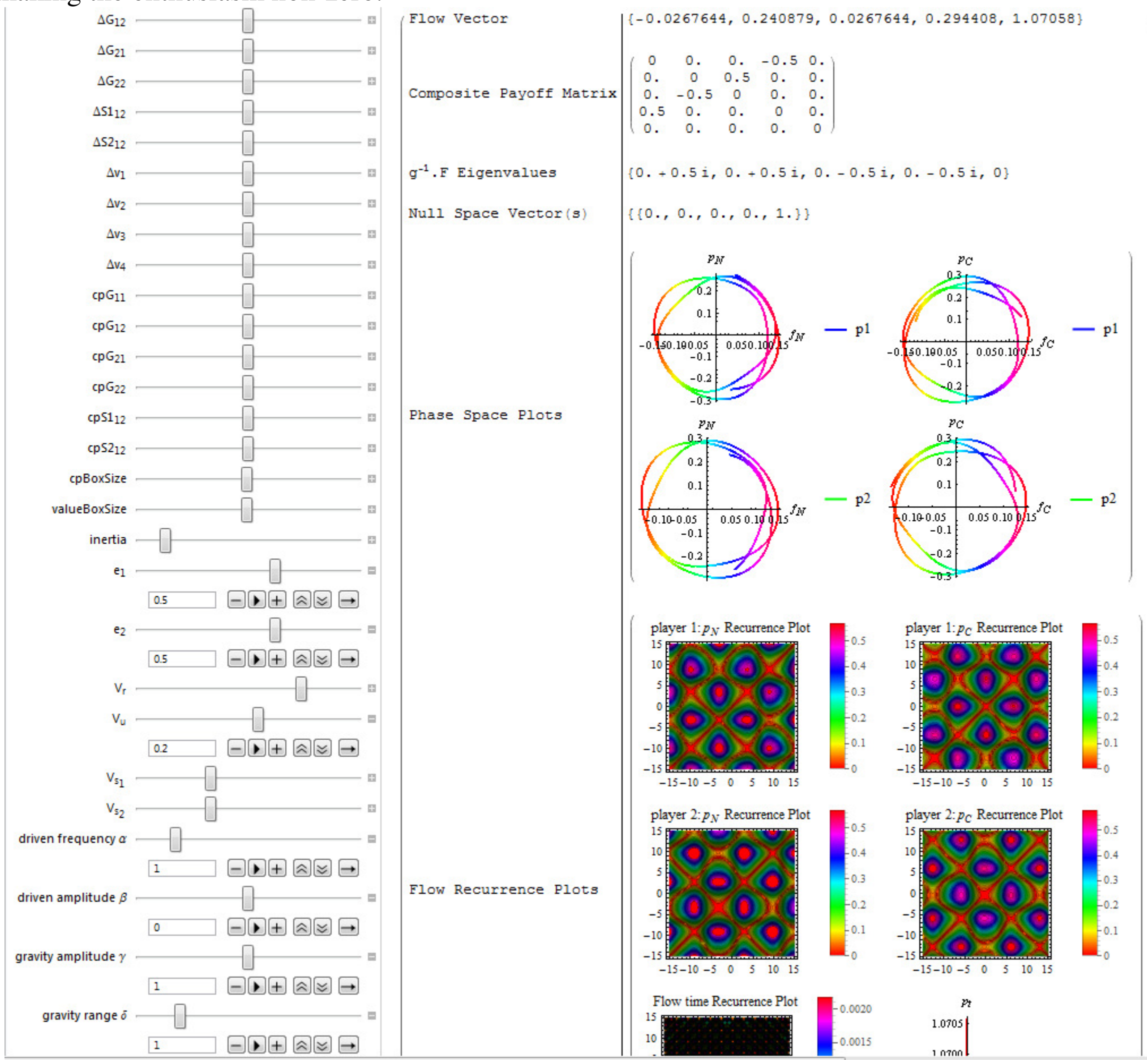

Figure 4: Prisoner's dilemma--space change

In Figure 4, we see the start of quasi-periodic behavior as a consequence of the non-linear space behaviors. You can see more by extending the proper time variable in the provided model. We also see that the recurrence plots are somewhat skewed, again an indication of possible quasi-periodic behavior.

In Figure 5, we see the slight deviations in the flow, and in the transformed flow variables that are directly related to the sum of the strategies, r, the difference or enthusiasm strategy u, and two remaining variables orthogonal to these that measure the relative weight or preferences between confess and not confess, $\{s_{1,} \quad s_2\}$. We are looking at "conserved" behavior because there are two conserved quantities: the total path length doesn't change and the total energy doesn't change for static fields. However, energy can change when the fields are dynamic; they can feed energy into or out of the system. This is the case as long as the time amplitude β is zero. We see both of these things within the numerical accuracy of the computation (Figure 5).

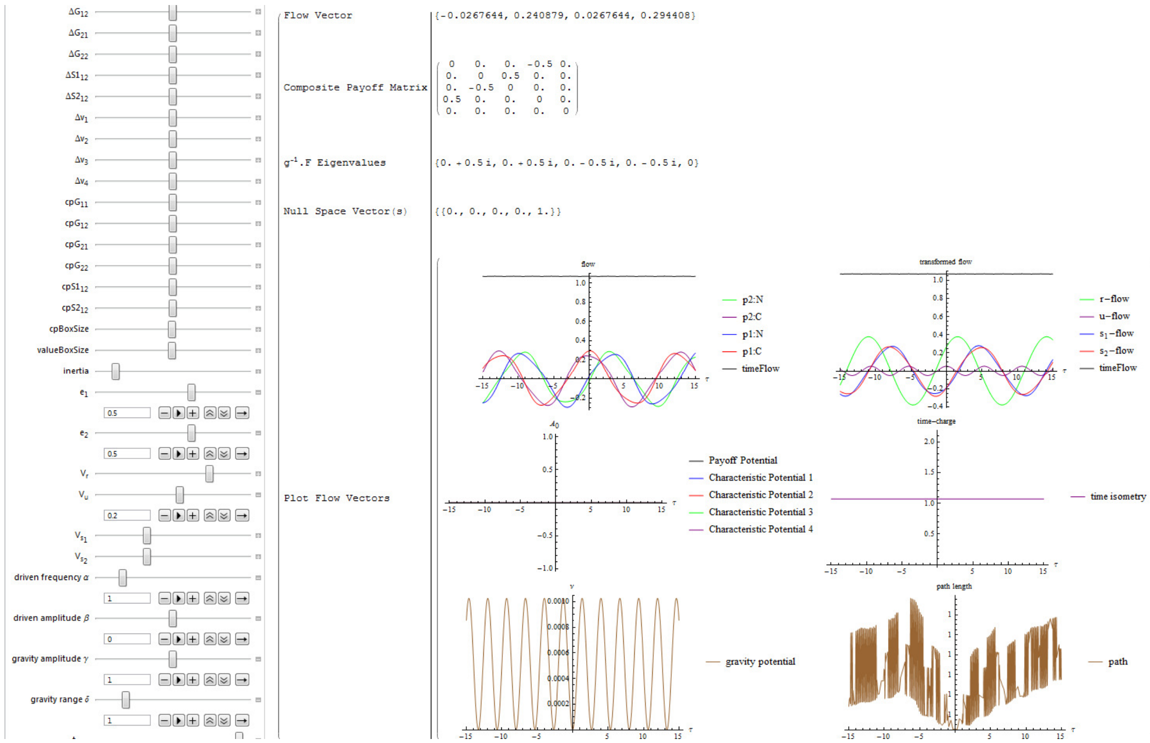

Figure 5: Detailed view

We also consider other contributions to the energy besides the flow: the composite payoff of the players. The "fifth" component of the payoff matrix is the gradient of the potential $A_0 = V_j A_0^j$, which is shown in the figure. It is zero (or constant), indicating that the fifth components are all zero. In decision processes, these components indicate the game value. In this baseline model, the game value is zero, another reflection of assuming that each player approaches the game in the exact same way.

Of course this is hardly likely. One player in general will have more interest than the other. We see the effect of making the interests slightly unequal, Figure 6, by changing the baseline values (of $\frac{1}{2}$ for e_k) for one of the players. In this example, we return the flow value to zero, so that we see the impact from one effect only.

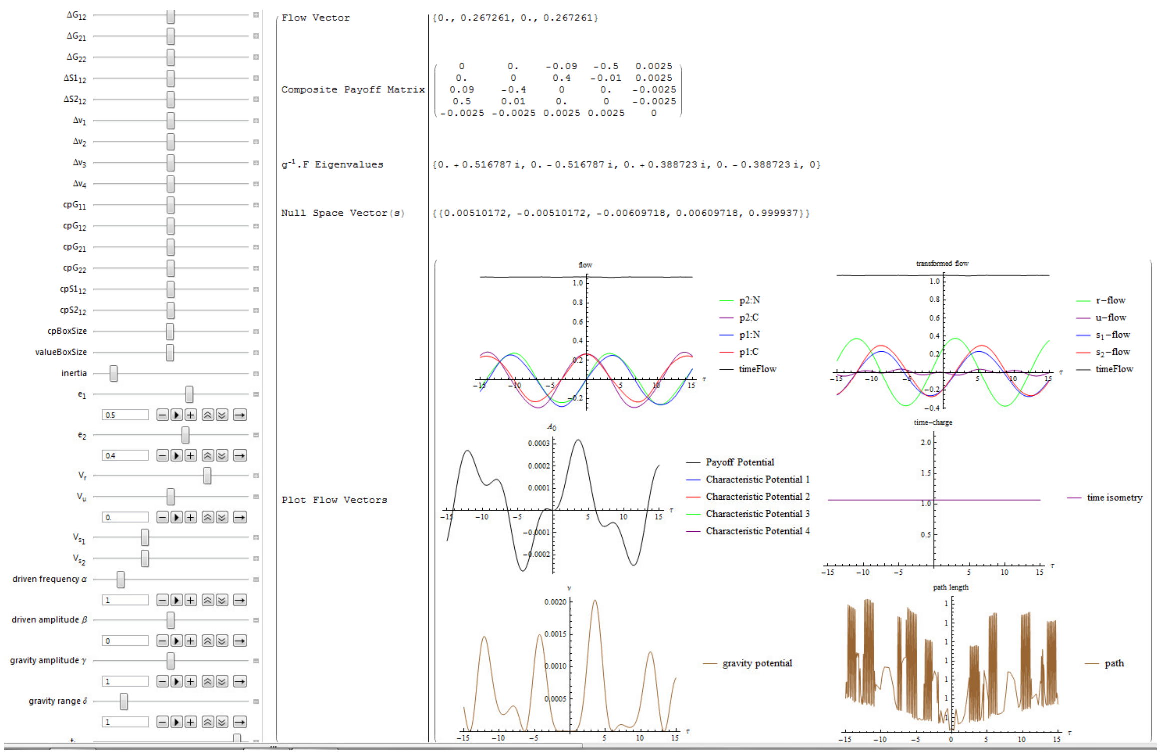

Figure 6: Baseline with interest difference

We see several changes.

- The null vector has changed.
- There is a game potential indicating the existence of a non-zero game value. The potential changes harmonically over time.
- The game potential along with the other dynamic effects result in a conserved “energy” and a conserved path length.
- Because of these changes, the enthusiasm flow is no longer zero. It is small but starting to make an effect.

The thing we notice is that what was a very bland set of behaviors to start with has generated structure with rather small changes in the underlying forces. So far, these changes are part of the conservative nature of the system and small changes caused only small effects. This is the first step in suggesting that much larger changes may lead to chaotic behaviors.

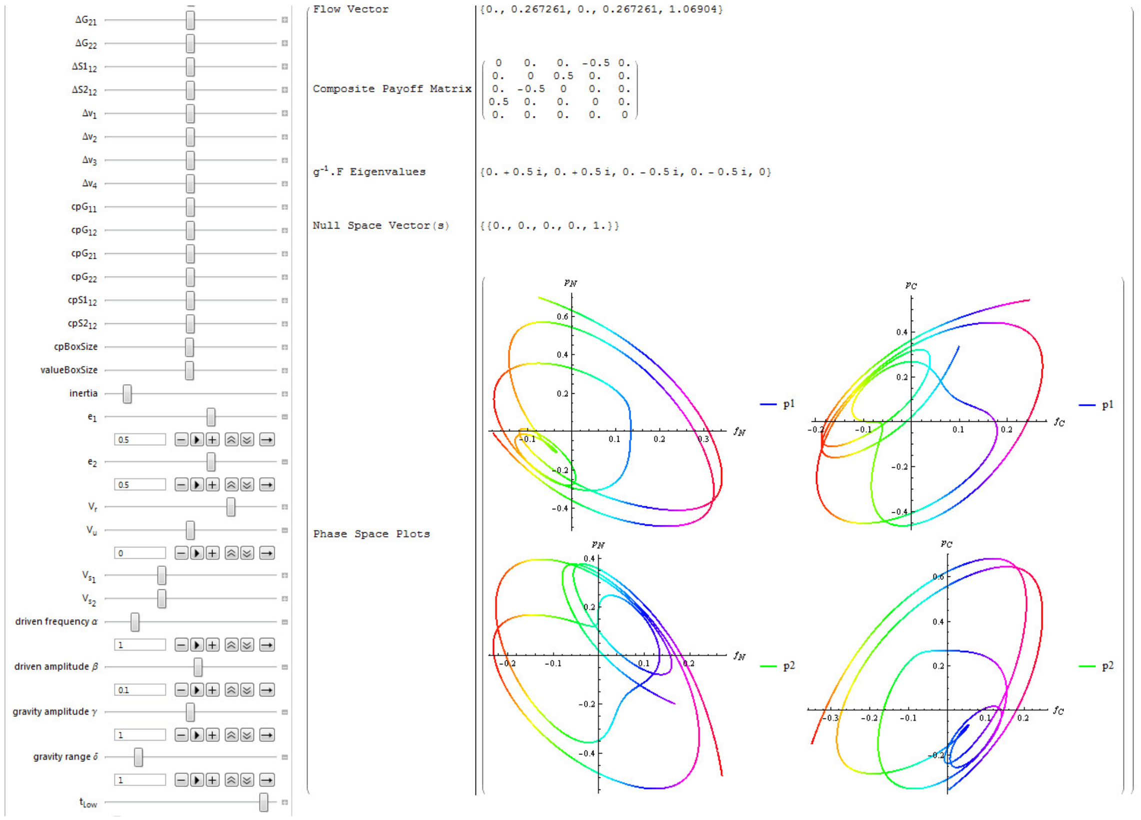

Figure 7: Small Driver amplitude

We return to the baseline and make a change in the driver amplitude, Figure 7. Small driver amplitude changes the periodic nature dramatically, in both the phase space plots and in the recurrence plots. If we focus on the time dependence structure of the phase space plots, Figure 8, we see that the flow structures changes.

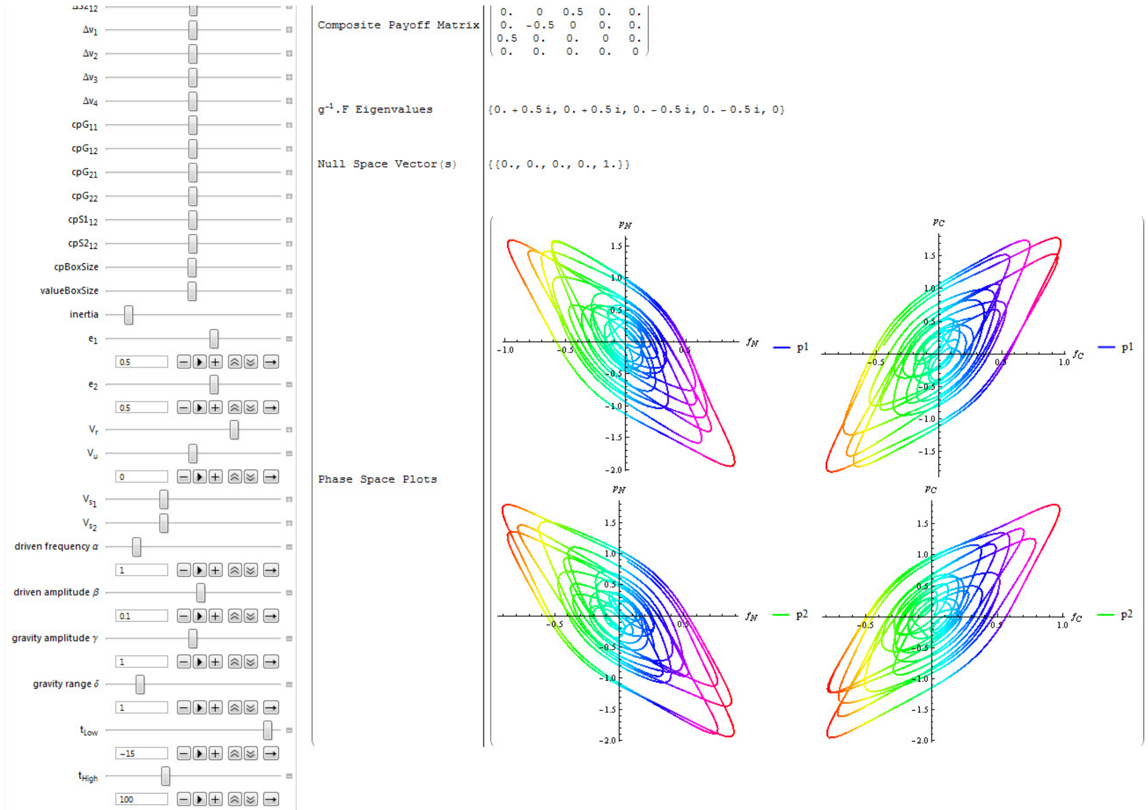

Figure 8: Small driver amplitude over longer time interval

We don't have a proof that this is chaotic behavior, but we start to see the types of structures we might expect: different attractor regions. To see this, we have expanded the time scale. We have returned to the case of no game value. Adding game value to this produces more variations. First, we start with the baseline on the expanded time scale, Figure 9.

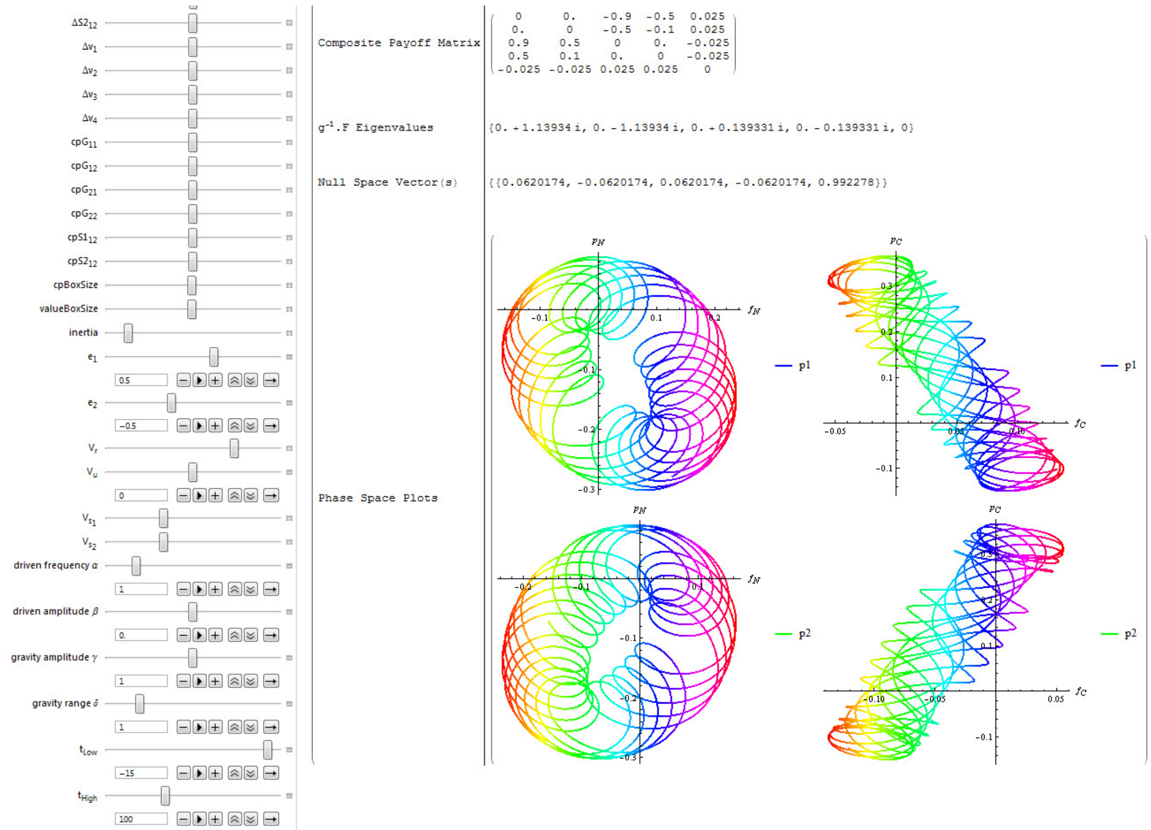

Figure 9: Opposite interests and no time driver

We see that the behavior is quasi-periodic with interlacing structures that may remind you of a donut structure. We have gone for a bigger effect by assuming one player has exactly the opposite interest of the other. In this case, note that the null space suggests that each player will not confess, again opposite to game theory and in this case one might anticipate that the behaviors are strong.

Now, the addition of the driver produces an even more profound effect, Figure 10.

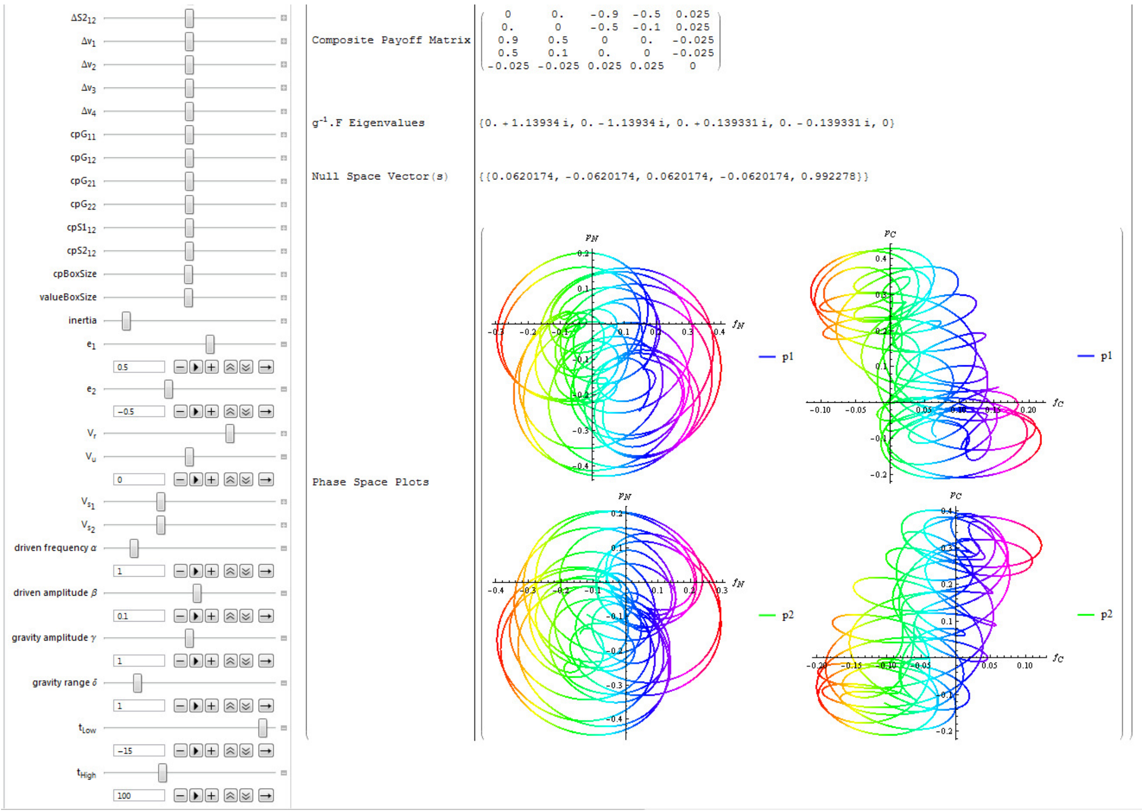

Figure 10: Opposite interests and time driver

The structure of the previous figure is still visible, but one would be tempted to call the resulting behavior chaotic or non-deterministic, Figure 11. Of course that would be incorrect, since these figures are generated from causal equations; it is just that the behaviors are sensitive to the initial conditions.

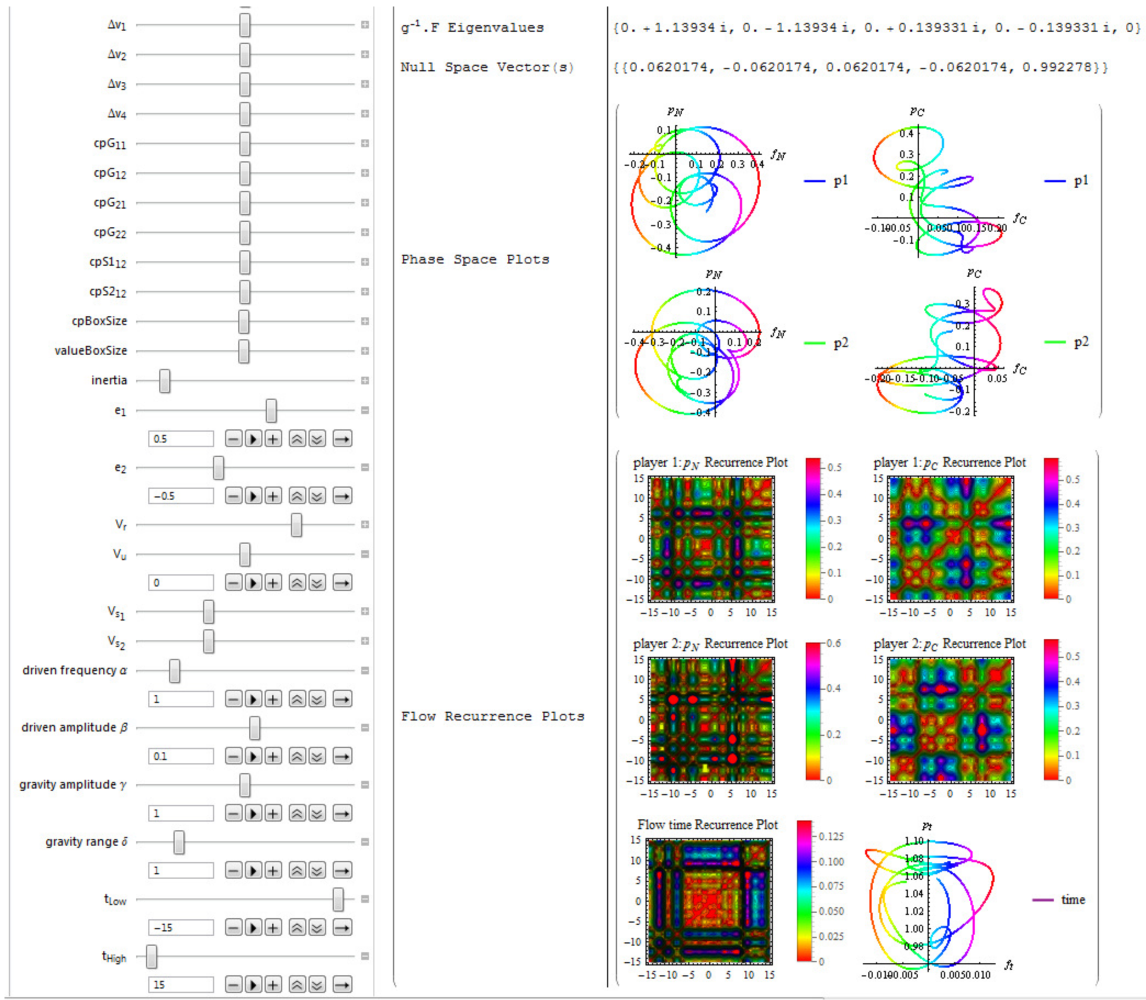

Figure 11: opposite interests and time driver

We have really only scratched the surface of the structures that are possible. We have not explored the effect of adding a ***characteristic potential*** that leads to new and interesting effects, such as Figure 12. We get "beats" like in radio transmissions. One key factor that produces this structure is the observation that the characteristic potential elements may have limits on their growth. This would occur, for example, if the potentials are bounded as any particular preference element grows. We capture this effect for both the *characteristic potentials* $\bar{A}_a^m$ and the value potential $e_j A_a^j$ with the following assumptions for the numerical exercise[1]:

$$
\begin{aligned}
A_0 &\equiv e_j A_0^j = L_v \sin\left(\pi \frac{v_a y^a}{L_v}\right) \\
\bar{A}_a^0 &= -\tfrac{1}{2} L_{cp} \sin\left(\pi \frac{f_{ab} y^b}{L_{cp}}\right)
\end{aligned}
\tag{B.7}
$$

[1] For those of you checking, with $\bar{g}^{ab} = -\delta^{ab}$ this choice for the characteristic potential satisfies the gauge $\bar{g}^{ab}\partial_a \bar{A}_b^0 = 0$ and the wave equation $\bar{g}^{cd}\partial_c \partial_d \bar{A}_a^0 = 0$. The choices here however are purely illustrative. A more exact treatment of *decision process theory* would derive potentials from the field equations (Thomas G. H., 2013, p. chapter 4). We would consider additional effects from the stress tensor as an example.

The value parameters v_a and box size L_v can be set by the sliders in the Mathematica notebook and are labeled Δv_a. They set the initial values. Similarly the initial values of the box size L_{cp} can be set by the sliders, as well as the *characteristic payoff* values f_{ab} that determine the initial values of the *characteristic potentials*. We notice from Figure 12 that the path length and "energy" remain conserved. The *characteristic payoff* captures a new mechanism for decision making: there can be a "drift" for making decisions that lies outside of the consideration of the decision process that nevertheless acts in the same way as a competitive force. It is not associated directly with a value in the way value is associated with each player. It thus has differences. Nevertheless it plays a crucial role.

A close analogy is the behavior of mechanics on the surface of the earth. In general, mechanics involves forces that impact motion of objects that obey Newton's laws. However, the earth is rotating; there will be effects due to this rotation above and beyond the normal mechanics. If you take a large enough view, these effects would have been included in any case. However, it is a great help in understanding to consider the effects piecemeal.

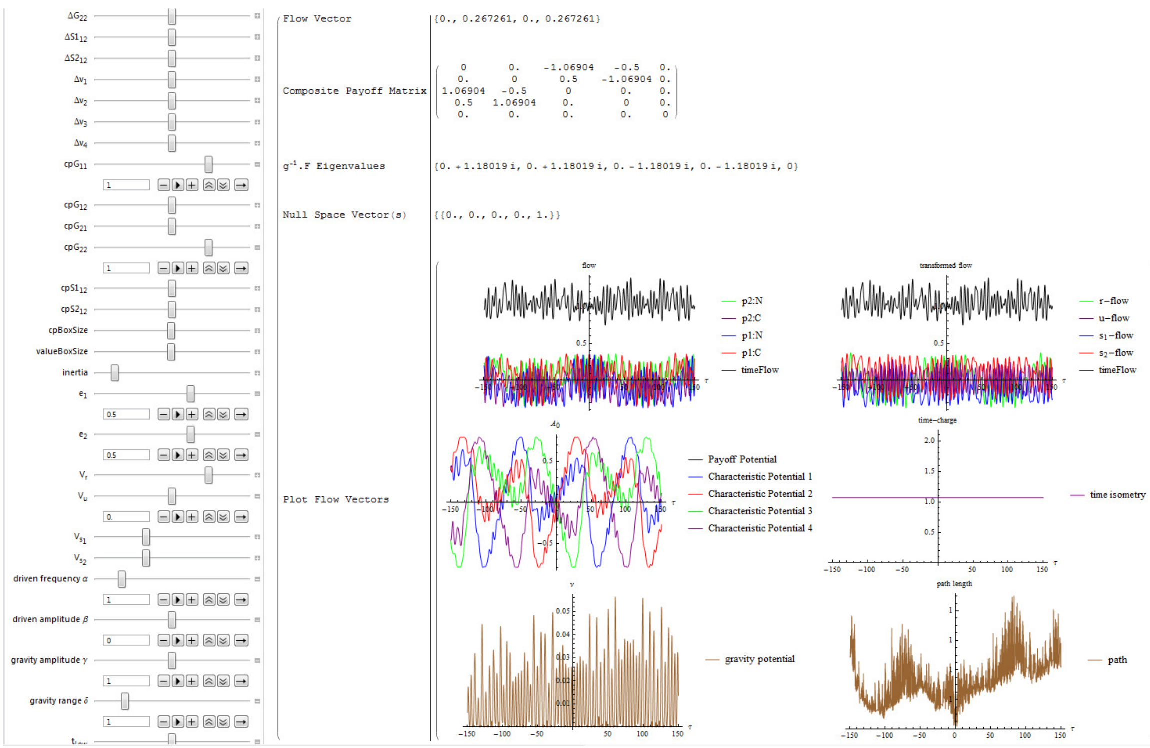

Figure 12: Add characteristic payoffs, labeled cpG_{mn}

B-4. Conclusions

So let's recap what we have learned from these numerical exercises. We have in the end generated behaviors from a *decision process theory* model that are on the one hand deterministic for frequencies, yet on the other hand, represent the uncertainty of choices based on these frequencies. We have separated out from the decision process the uncertain aspect of the decision, whose future behavior is unknown: we don't know which pure strategy will be chosen. Thus we have identified that aspect of the decision process that deprives us of perfect information. We have also identified those aspects of decisions processes that might evolve continuously in time and can be determined in a causal manner. These are the numerical frequencies of choice that form the basis of the choice, but don't actually determine the specific choice at any given time. Our theory is then about the frequencies and not about the choices. There is reason to believe that such frequencies and the forces that govern them represent something invariant about the decision process.

Because of chaotic behaviors, this is not the end of the story. Just because we have a theory that determines future behavior based on knowledge of past behavior, we are not justified in assuming that the predications will be insensitive to our starting point. Only non-chaotic behavior assumes that the future behavior is not sensitive to small changes in the starting point. This often follows from theories that are linear in nature. By contrast chaotic behavior expects small behaviors to generate large behavioral differences, even if that behavior ultimately stays bounded. That is in fact what we see above; the behavior stays bounded and roughly follows the non-chaotic behaviors. Over time however, we see significant deviations. Some of the non-linear behavior is a consequence that preferences can't grow without limits. We have postulated that concept here, but in fact do see evidence for that behavior in the full theory, (Thomas G. H., 2013).

We also see that chaotic behaviors can be generated from within, without recourse to external "drivers', if there are suitable parameters that can be varied. For the pendulum, the suitable variable would be the initial speed. The initial flows and payoffs are suitable variables for decision processes. The chaotic behavior can also be generated from a "driver" representing external periodic forces. It is then a matter of whether the amplitudes and frequencies excite the underlying structures. In fact from both, we see that seemingly benign behaviors as a steady state need not indicate the lack of interesting structures. The key is how to excite these structures into existence. The CDF and notebook files are provided to facilitate future investigations by the reader.

References

Oxford University. (2009). Oxford English Dictionary Second Edition on CD ROM (v4.0). *OED* . Oxford, England: Oxford University Press.

Thomas, G. H. (2006). *Geometry, Language and Strategy.* New Jersey: World Scientific.

Thomas, G. H. (2013). *The Dynamics of Decision Processes.* Retrieved from http://www.decisionprocesstheory.com

Thomas, G. H., & Kane, K. (2008). Physical Decision Theory, the Prisoner's Dilemma and a New Foundation for Nash Equilibrium. *unpublished* .

Von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior.* New York: John Wiley & Sons, Inc.