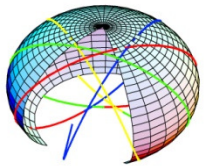

Visualizing behaviors in differential geometries

Jerry Thomas
Milwaukee School of Engineering
Contact information at http://decisionprocesstheory.com/

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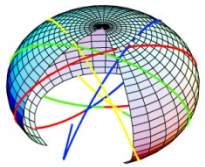
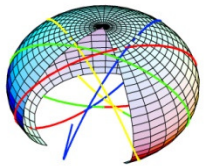
University of Illinois—Chicago, IL



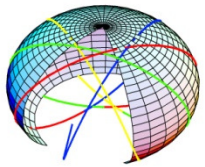
Abstract

- I have been investigating a class of geometries that arise out of physics and engineering that also may also provide extensions and generalizations of theories to the social sciences. Specifically, theories I have been looking at are, to use an analogy from electrical engineering, extensions from DC behavior to AC steady state and AC transient behaviors. In geometric terms, the extensions are from flat space to locally flat spaces with curvature in space and time. What comes out of these investigations are a number of interesting visualizations. I present some of these visualizations along with an introduction about where they come from. I discuss to what extent these behaviors are possibly chaotic.



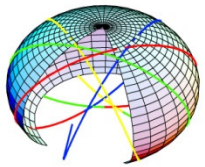
What is it I want to "see"?

- Engineering Disciplines help us "see"
  - Measurements
  - Physical Laws
  - Mathematical
- Examples
  - Electrical
  - Mechanical
  - Industrial



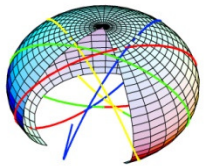
Local versus global concepts

- Locally—the good, easy to "see"
  - Build small things—blocks for example
  - Temporary structures
  - No engineering required
- Globally—the bad, hard to "see"
  - Large structures—towers for example
  - Permanent structures
  - Past history helpful
  - Engineering required



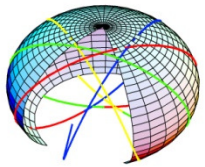
Engineering attributes

- Extrapolating from local to global continuously
- Mathematical discipline of calculus (hence topology and differential geometry)
- Requires quantitative measurements about the local to make credible extrapolations

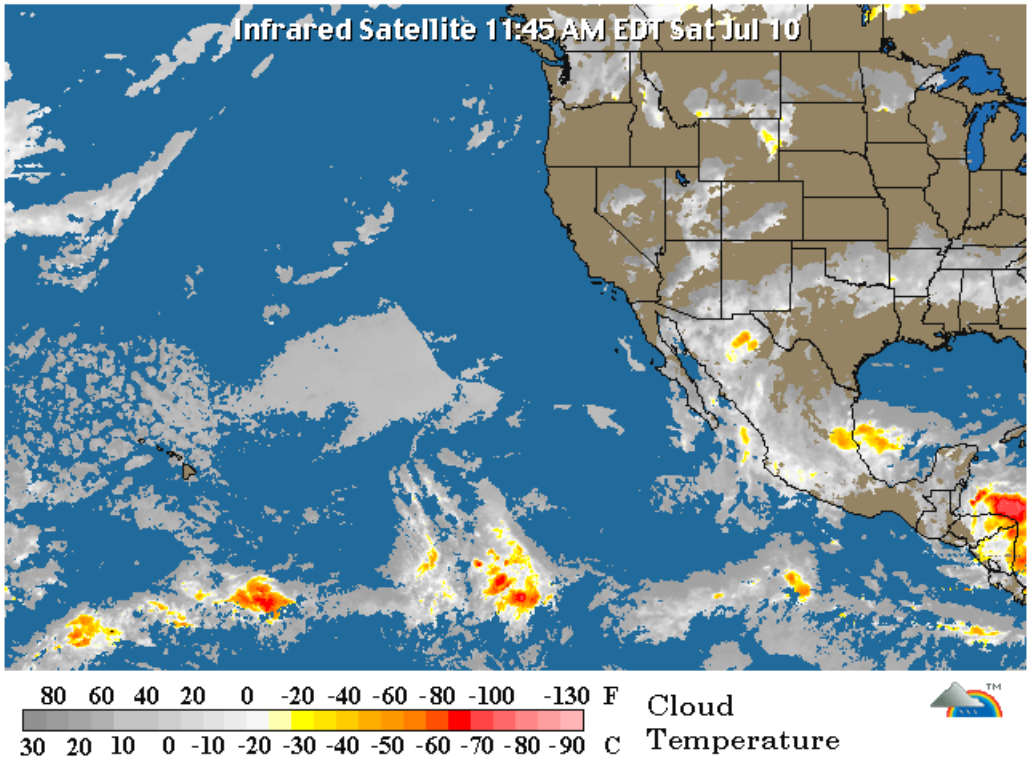


Meteorology as an example

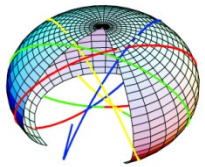
- Our ordinary view: Farmer's Almanac has limited vision. Earth is flat, not moving.
- Modern view: Take a snapshot of the wind, pressure and temperatures over the earth. Include Coriolis.
- What will these values be in an hour? A day? A month?
- Use local behaviors from fluid mechanics to provide differential properties
- Integrate these properties over time to produce the result



Snapshot

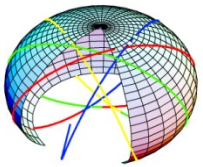

Infrared Satellite 11:45 AM EDT Sat Jul 10
80 60 40 20 0 -20 -40 -60 -80 -100 -130 F
30 20 10 0 -10 -20 -30 -40 -50 -60 -70 -80 -90 C
Cloud Temperature

Flat Space, DC Currents and Game Theory

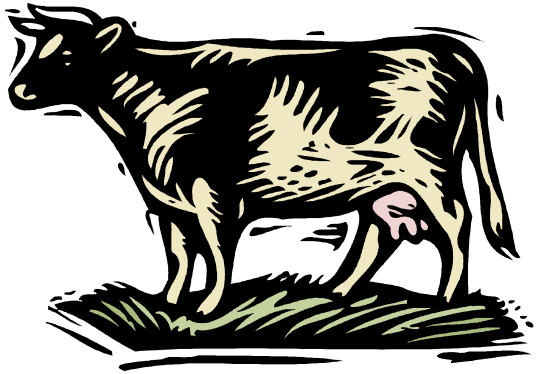


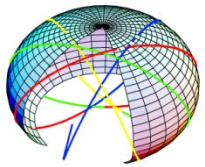
- Conceptually—flat means nothing changes
- DC currents—Currents are constant
- Start from Physics: Newtonian mechanics and rigid bodies (why is this flat?)

$$\mathbf{F}_k = m_k \ddot{\mathbf{r}}_k$$



Rigid Body, constraints or inactive?

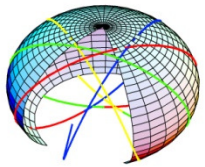




Lagrange approach to all three

- Provide a "simple" way to include ALL forces
- Yet maintain a focus on only the active variables
- Include "inactive" in energy-momentum
- Start with momentum of every component

$$\mathbf{F}_k = \dot{\mathbf{p}}_k$$

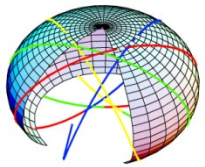


Lagrange's Method

- Eliminate the constraints
- Variations change only active variables
- The constraints hold other variations to zero

$$\sum_k \left(\mathbf{F}_k - \dot{\mathbf{p}}_k \right) \delta \mathbf{r}_k = 0$$

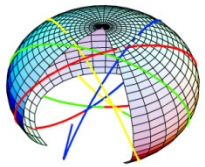
$$\sum_k \left(\mathbf{F}^{(a)}{}_k - \dot{\mathbf{p}}_k \right) \delta \mathbf{r}_k = 0$$



Lagrange's method (II)

- Rewrite product of force times distance as kinetic energy T
- Write result in terms of active variables q

$$\left\{\begin{aligned} T &= \tfrac{1}{2}\sum m_k \dot{\mathbf{r}}_{\mathbf{k}}^{\,2} \\ Q_\alpha &= \sum \mathbf{F}^{(\mathbf{a})}_{\mathbf{k}} \cdot \frac{\partial \mathbf{r}_{\mathbf{k}}}{\partial q_\alpha} \end{aligned}\right\} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

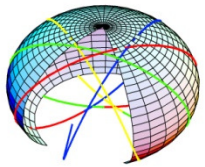


How to see physics as flat

- For important class of problems, the forces can be expressed in terms of potentials
- Final result has geometric interpretation: ***principle of least action ("the new flat")***

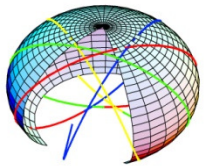
$$F^{(a)}{}_k = -\frac{\partial V}{\partial r_k} + \frac{d}{dt}\left(\frac{\partial V}{\partial v_k}\right) \Rightarrow Q_\alpha = -\frac{\partial V}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right)$$

$$L = T - V \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \Leftrightarrow \delta \int_{path} L dt = 0$$



Recap—physics as geometry

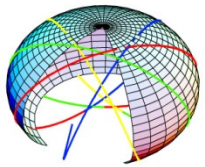
- Principle of least action reduces problems of many variables to problems with fewer variables
- Must express full kinetic energy and potential energy in terms of the new (active) variables
- Lagrange restates Newtonian mechanics as a geometric "least path" or "flat"



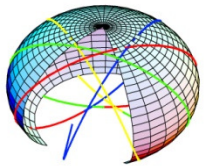
Einstein physics

- The world is flat
  - Well not globally but locally
- Einstein noted that electromagnetic theory has a different local property from Newtonian physics
  - Maxwell was right; Newton was wrong
  - Signals travel at the speed of light (always)
  - No rigid bodies; but inactive isometries
  - Coordinates are potential fields (like voltages)
- The modern view of the world: it is locally flat
  - Dynamics still governed by least action
  - Locally flat—space time is a ***fibre bundle***

What about decisions? Are they flat?



- Are there equilibrium or non-changing situations?
- Create a snapshot
  - what do we need to know?
  - What are the "spatial" variables?
  - Over what region?
- What are the laws governing the extrapolation? Dynamics?
  - Look to economic and social theories
  - Fix on game theory as a basis for "flat" (measurements=utility function)
  - Game theory is a dynamic! Equilibrium



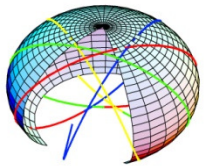
Game Theory

- Game Theory provides space-time in terms of strategies and physical time
- Analysis reduces the "intrinsic form" possibilities to matrices of numbers, one for each player
- E.g. prisoner's dilemma

G^1_{12}	N_2	C_2
N_1	–0.1	–1
C_1	0	–0.9

and

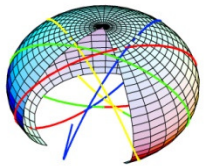
G^2_{21}	N_1	C_1
N_2	–0.1	–1
C_2	0	–0.9



Plug for my book

- The approach starts with the Theory of Games, but departs with significant deviations and help from physical and engineering theories.
- These ideas appeared first in Kauffman's knot series: Thomas, G. H. (2006), *Geometry, Language and Strategy*, New Jersey, World Scientific
- Volume 2: Thomas, G. H. (in preparation), *The Dynamics of Decision Processes*
- *Both* with *Partnership Program Support* from Wolfram Research.

Electricity and Magnetism as Game Theory

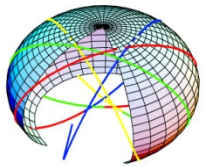


- We don't use physics as just an analogy but as a way to relate game theory to differential geometry
  - Gauge theories imply No monopoles and Faraday
  - Gauge theories have Source equations—Coulomb and Ampere
- Force equation—identify field with payoffs

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

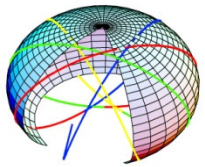
$$\partial_c F_{ab} g^{bc} = \frac{1}{c}\rho g_{ab} V^b \Leftrightarrow \partial_b F^{ab} = \frac{1}{c}\rho V^a$$

$$m\frac{dV^a}{d\tau} = eF^a{}_b V^b$$



DC Currents

- For constant flow the time rate of change of the flow is zero
- As a consequence the flow must be in the null space of the field matrix
  - This is the statement of "equilibrium" in game theory
  - "statics" is game theory; special case of least action

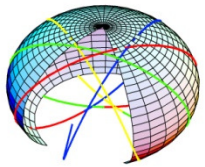


AC Steady State--resonance

- AC Steady-state behavior can be introduced by assuming that the flow is oscillating
- The flow equation now has solutions for specific eigen-frequencies—solutions in volume 1

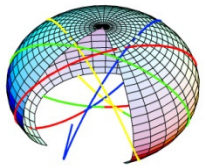
$$m\frac{dV^a}{d\tau} = eF^a{}_b V^b, \quad V^a = \bar{V}^a e^{i\omega t} \Rightarrow \quad i\omega \bar{V}^a = \frac{e}{m} F^a{}_b \bar{V}^b$$

AC steady-state forced behaviors



- Non-resonant and transient behaviors result looking at the complete solution of the equations
- Take seriously the geometry of decisions:
  - One dimension for each active strategy
  - One dimension for each inactive strategy, which is a local symmetry of the geometry
  - One dimension for time
- Inactive symmetries induce electromagnetic fields
- Rewrite geometry Using the theory of forms:

$$\mathbf{A} \equiv A_a dx^a$$

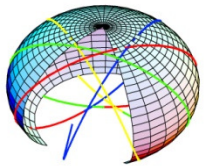


Principle of Least action (PLA)

- Tensors are the geometric structures
  - Determined from acceleration potentials
- Potentials carry energy and momentum
- The PLA determination in terms of forces is a dynamic theory, relates curvature to the "all":

$$\mathcal{R} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}$$

$$\bar{R}_{ab} - \tfrac{1}{2} g_{ab} \bar{R} = -\kappa T_{ab} - \kappa T_{ab}^{payoff} - \kappa T_{ab}^{inactive}$$

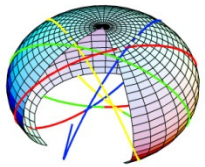


Geometry and least paths

- The analogy with AC and DC relies on the identification of "electromagnetic fields" in higher dimensions—consequence of symmetries however
- Stokes Law
  - No monopoles
  - Faraday's law

$$\mathbf{F} = d\mathbf{A} = F_{ab} dx^a \wedge dx^b$$

$$\oint_{\partial K} \mathbf{F} = \int_K d\mathbf{F} = 0$$



Leads to source equations

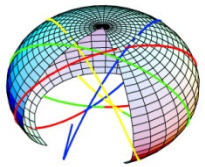
- Current Density
- Dual Electromagnetic form
- Source equation (generalizes Maxwell)

$$*\mathbf{j} = \varepsilon_{ab\cdots f}\, j^a dx^b \wedge \cdots \wedge dx^f$$

$$*\mathbf{F} = \varepsilon_{a\cdots def}\, F^{ef} dx^a \wedge \cdots \wedge dx^d$$

$$d * \mathbf{F}/\kappa = *\mathbf{j}$$

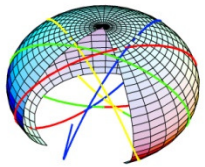
$$\oint_{\partial K} * \mathbf{F}/\kappa = \int_K d * \mathbf{F}/\kappa = \int_K *\mathbf{j}$$



Hidden dimensions

- Symmetry leads to hidden dimensions
- They are a type of constraint
- Lagrange approach: rewrite metric (all measurement potentials) in terms of active variables only
  - Hidden dimensions contribute to the energy momentum
  - Hidden dimensions contribute metric elements but no contribution to the metric element functional behaviors
  - Line element will then have "e-m" potentials (Kaluza-Klein)

$$ds^2 = \gamma_{jk}\left(d\xi^j + A_a^j dx^a\right)\left(d\xi^k + A_b^k dx^b\right) + g_{ab}dx^a dx^b$$

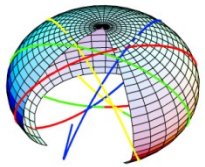


Normal form basis

- Coordinate Basis for active dimensions
- Not a coordinate basis for inactive

$$\mathbf{U}^a = dx^a$$

$$\mathbf{U}^j = d\xi^j + A_a^j dx^a$$



Geodesic flows in this space

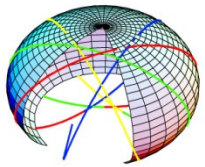
- Associate game theory flows as follows
  - Active and inactive respectively

$$V^a = \frac{dx^a}{d\tau}$$

$$V^j = \frac{d\xi^j}{d\tau}$$

$$\frac{DV^a}{\partial\tau} \equiv V^a{}_{;b} V^b = 0$$

$$\frac{DV^j}{\partial\tau} \equiv V^j{}_{;a} V^a = 0$$



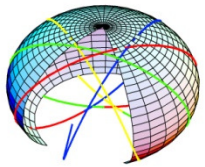
Covariant flow equations

- Active component flows generalize game theory
- Payoff is an acceleration effect: hidden vorticity
- Acceleration frame effects—no longer statics

$$\frac{DV^a}{\partial\tau} = V^a{}_{;b}\frac{dx^b}{d\tau} + V^a{}_{;j}\frac{d\xi^j}{d\tau} = V^a{}_{;b}V^b + V^a{}_{;j}V^j = 0$$

$$\frac{dV^a}{d\tau} + \varpi^a{}_{cb}V^cV^b + \varpi^a{}_{jb}V^jV^b + \varpi^a{}_{bj}V^bV^j + \varpi^a{}_{kj}V^kV^j = 0$$

$$g_{ab}\frac{dV^b}{d\tau} + \varpi_{abc}V^bV^c = V_kF^k_{ab}V^b - \tfrac{1}{2}V_jV_k\partial_a\gamma^{jk}$$



Persistency

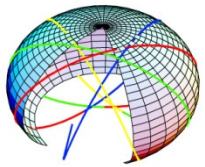
- Inactive flows are persistent (conserved)

$$\frac{DV^j}{\partial\tau} = V^j{}_{;a} V^a + V^j{}_{;k} V^k$$

$$\frac{DV^j}{\partial\tau} = \frac{dV^j}{d\tau} + \varpi^j{}_{cb} V^c V^b + \varpi^j{}_{kb} V^k V^b + \varpi^j{}_{ck} V^c V^k = 0$$

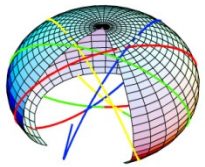
$$\frac{dV^j}{d\tau} + \gamma^{jl} \frac{d\gamma_{lk}}{d\tau} V^k = 0$$

$$\frac{dV_k}{d\tau} = 0$$



In a differential geometry

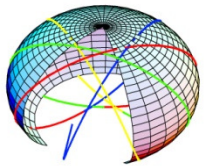
- Each symmetry has associated with it:
  - A "Coriolis" force
  - A gradient "centripetal" force
  - A conserved momentum
- In decision process theory
  - A player is associated with a symmetry
  - Therefore the payoffs are the "Coriolis" rotations



AC Steady-state behaviors

- Just as in EE, this behavior requires us to solve in more detail the underlying equations.
- We have to determine coordinate potentials
  - Harmonic gauge: in our models, coefficients are independent of time
- What are the key equations?
- In 3+1 dimensions: the harmonic gauge wave equation (with explicit time isometry potential) is:

$$\overline{g}^{\overline{\tau}\overline{\tau}}\overline{\partial}_{\overline{\tau}}{}^{2}x^{a}-\nabla^{2}x^{a}+2\mathbf{A}\bullet\nabla\overline{\partial}_{\overline{\tau}}x^{a}+\left(\nabla x^{a}-\mathbf{A}\overline{\partial}_{\overline{\tau}}x^{a}\right)\bullet\nabla\left(q+\Theta\right)=0$$



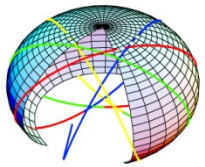
Tricks from AC Circuits

- It may be noticed that solving the equations in the co-moving frame is again a type of DC circuit: (like circuit theory)
- Harmonic wave equation is linear! Time behavior is not!!
- One of the tricks of AC Steady State circuits is the use of phasors to eliminate the (proper) time dependence
- That can be done with the harmonic wave equation
- These solutions are steady-state AC solutions
- Note the equations are linear in coordinate potentials but non-linear in the co-moving parameter space

$$x^a = \overline{x}^a e^{i\omega\tau}$$

$$-\omega^2 \overline{g}^{\overline{\tau}\overline{\tau}} \overline{x}^a - \nabla^2 \overline{x}^a + i2\omega \mathbf{A} \bullet \nabla \overline{x}^a + \left(\nabla \overline{x}^a - i\omega \mathbf{A} \overline{x}^a \right) \bullet \nabla \left(q + \Theta \right) = 0$$

Numerical AC Steady-State Solutions



- Many equations, many variables and hence many interesting phenomena
- Here is an example of a gauge condition constraint—chaotic?

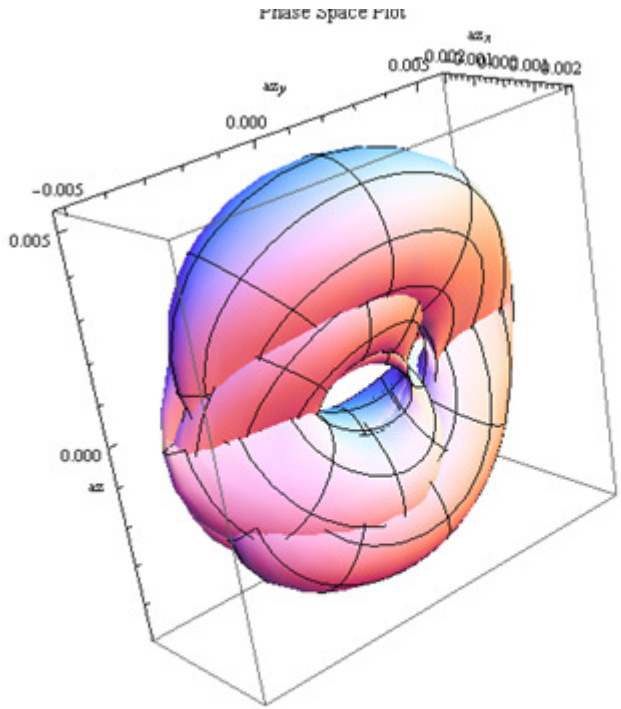
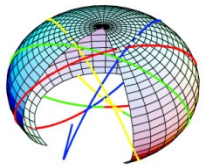
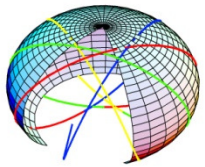


Figure 9-35: Gauge Condition Surface $mesh = 6$ **for:** $\partial_x a_z \quad \partial_y a_z \quad a_z$



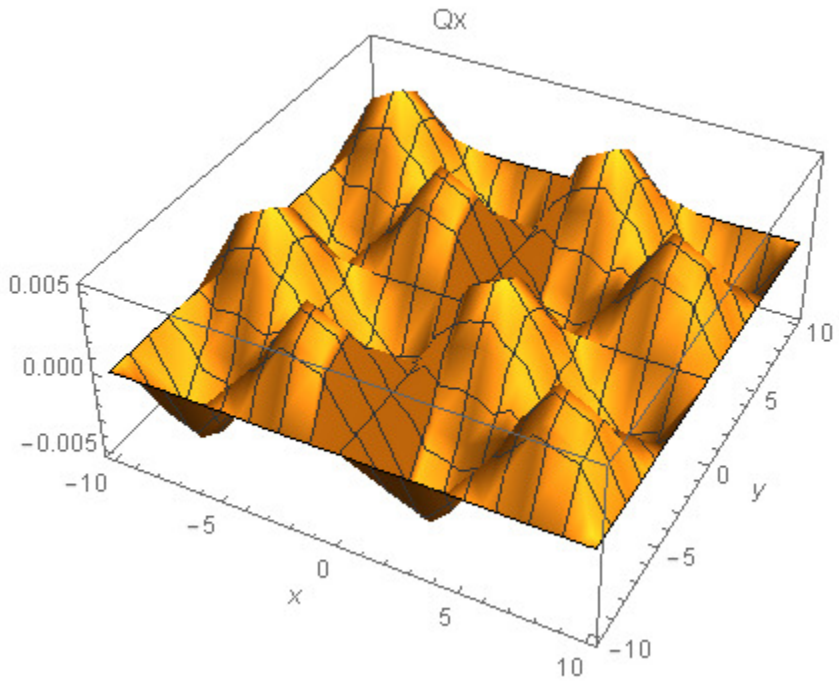
Resonance and acceleration

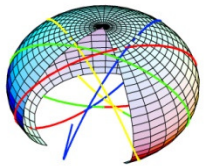
- One question that may occur to you is whether you see the discrete eigen-frequency solutions in the general AC steady state case
- It is perhaps like finding such solutions in the transfer functions of electrical engineering
- Such solutions do exist and occur when the local space is flat and the acceleration is small



Harmonic solution—volume 2

- Here is the acceleration from the numerical computations near where the space is flat (as set by initial conditions)
- The other two components of acceleration are comparable or smaller
- *Cf.* Chapter 8, exercise 17 and see that the harmonic in question is the transformation, not the harmonics in the figure





Frame Streamlines—volume 2

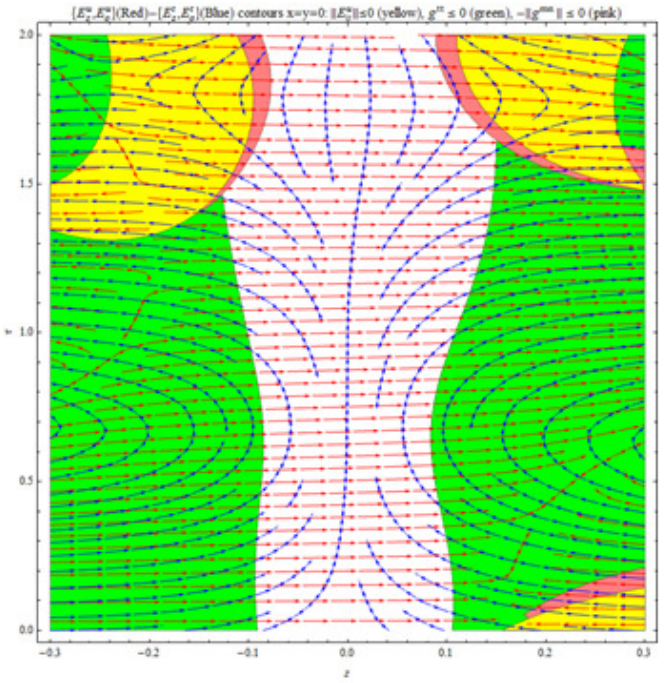
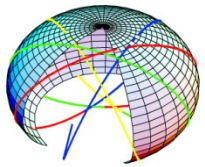


Figure 9-37: Frame Streamlines for three harmonic model



Communication—volume 2

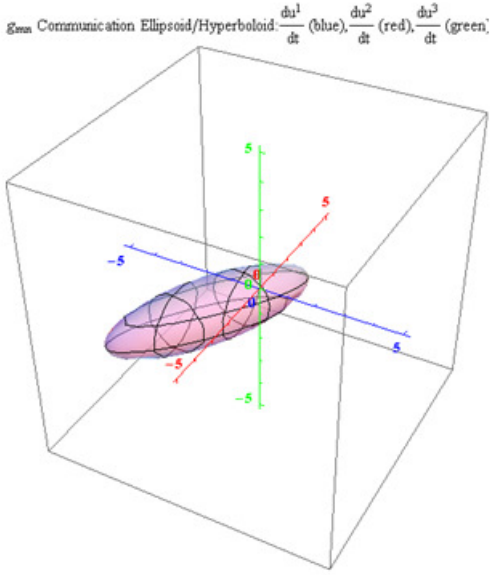


Figure 10-17: Communication ellipsoid $g_{tt} > 0, \quad g^{tt} > 0$

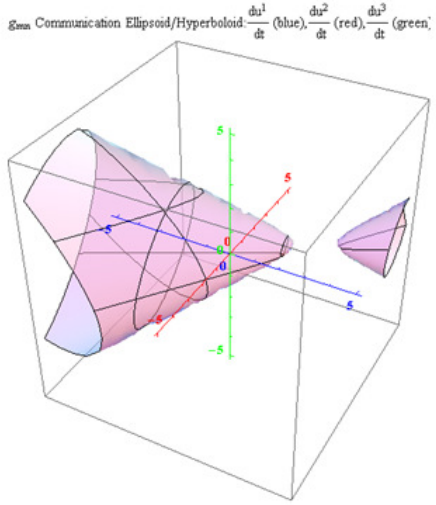
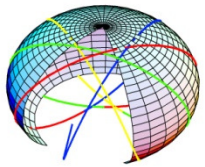


Figure 10-18: Communication hyperboloid $g_{tt} > 0, \quad g^{tt} < 0$



Time and "beta"—volume 2

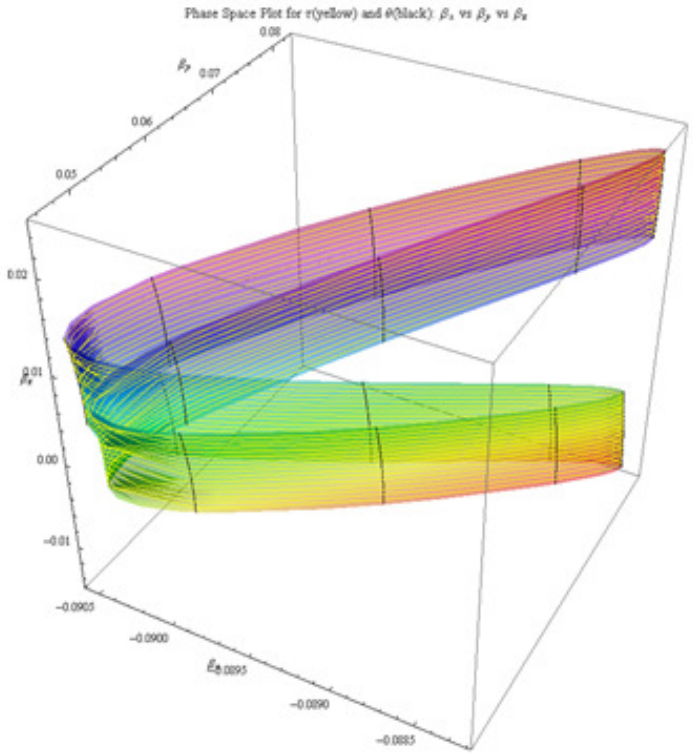
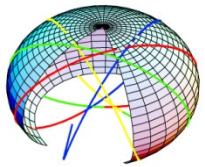
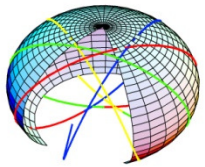


Figure 10-23: Now-β for the two-harmonic model along u



Solution strategy for transients

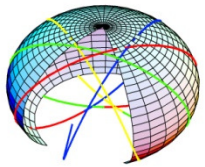
- We now have DC solutions (co-moving frame)
- We now have AC Steady-State solutions (phasors)
- To get transient solutions, note that the Differential geometry
  - Is a gauge theory
  - Is set by picking the harmonic gauge
  - Is set by the reference frame in which the metric can be determined
    - I pick the co-moving frame
  - Is determined by the equations for the coordinates in a general frame with given initial and boundary conditions



Very interesting model is 1+1

- Consider 1+1 dimension for illustration
- Although the parameters of the equation can be solved numerically, let's take them as given functions
- We will choose forms for them consistent with what has been found from the numerical solutions to the harmonic wave equation

$$\left(1+e_{\alpha}e^{\alpha}\right)\mathrm{e}^{2q}\,\overline{\partial}_{\overline{\tau}}{}^{2}\overline{x}^{a}-\left(q^{\upsilon}+\omega^{\upsilon\alpha}{}_{\alpha}\right)\overline{\partial}_{\upsilon}\overline{x}^{a}-\overline{\partial}_{\upsilon}\overline{\partial}_{\upsilon}\overline{x}^{a}=0$$



Numerical approximations

- Here is what I get

$$-sizeZ \le z \le sizeZ$$

$$\hat{z} \equiv \frac{z}{sizeZ}$$

$$\psi \equiv 1 + e_\alpha e^\alpha = \mathrm{e}^{-\beta \hat{z}^2}$$

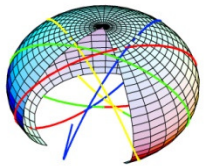
$$q \equiv -\chi \frac{\hat{z}^2}{q_d + \hat{z}}$$

$$q^z = -\partial_z q$$

$$\Theta \equiv \omega^{v\alpha}{}_\alpha = \omega^{z\alpha}{}_\alpha = 5\,\mathrm{e}^{-\frac{1}{4}\alpha(1+\hat{z})^2}$$

$$\left(1 + e_\alpha e^\alpha\right)\mathrm{e}^{2q}\,\overline{\partial}_{\overline{\tau}}{}^2 \overline{x}^a - \left(q^z + \omega^{v\alpha}{}_\alpha\right)\overline{\partial}_v \overline{x}^a - \overline{\partial}_v \overline{\partial}_v \overline{x}^a = 0$$

$$\psi\,\mathrm{e}^{2q}\,\overline{\partial}_{\overline{\tau}}{}^2 \overline{x}^a - \left(q^z + \Theta^z\right)\overline{\partial}_z \overline{x}^a - \overline{\partial}_z{}^2 \overline{x}^a = 0$$



Frame Transformations

- Rethink what it means to be a local coordinate
- You can have a familiar frame that is co-moving
- Frame transformations are needed however in a general frame

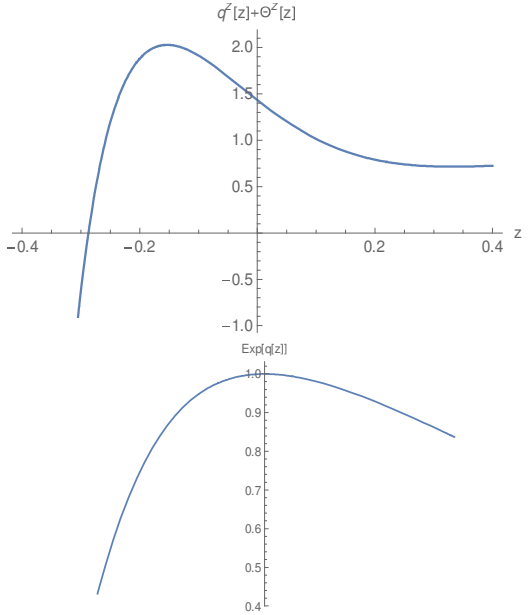
$$\psi \mathrm{e}^{2q} \bar{\partial}_{\bar{\tau}}{}^{2} \bar{x}^{a} - \left(q^{z} + \Theta^{z}\right) \bar{\partial}_{z} \bar{x}^{a} - \bar{\partial}_{z}{}^{2} \bar{x}^{a} = 0$$

$$x^{a}\left(z,\tau\right) = \bar{x}^{a}\left(z,\tau \mathrm{e}^{q}\right) = \bar{x}^{a}\left(z,\bar{\tau}\right)$$

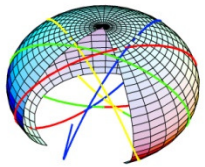
$$\bar{\tau} = \tau \mathrm{e}^{q}$$

$$E^{a}{}_{o} = \mathrm{e}^{q} \bar{\partial}_{\bar{\tau}} \bar{x}^{a}$$

$$E^{a}{}_{z} = \partial_{z} \bar{x}^{a}$$



Sources: Boundary/Initial Conditions

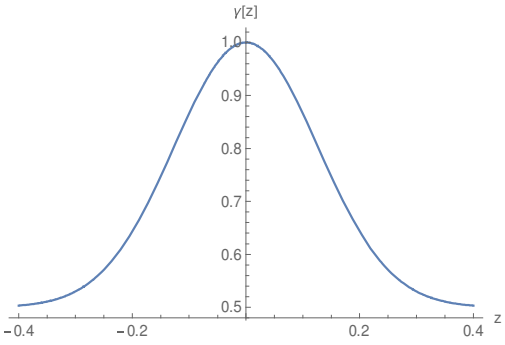


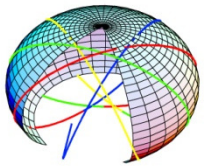
- Sources
  - These are Boundary Conditions
  - These are also Initial Conditions

$$t(z,\bar{\tau}) = 0 + \gamma(z)\bar{\tau}$$

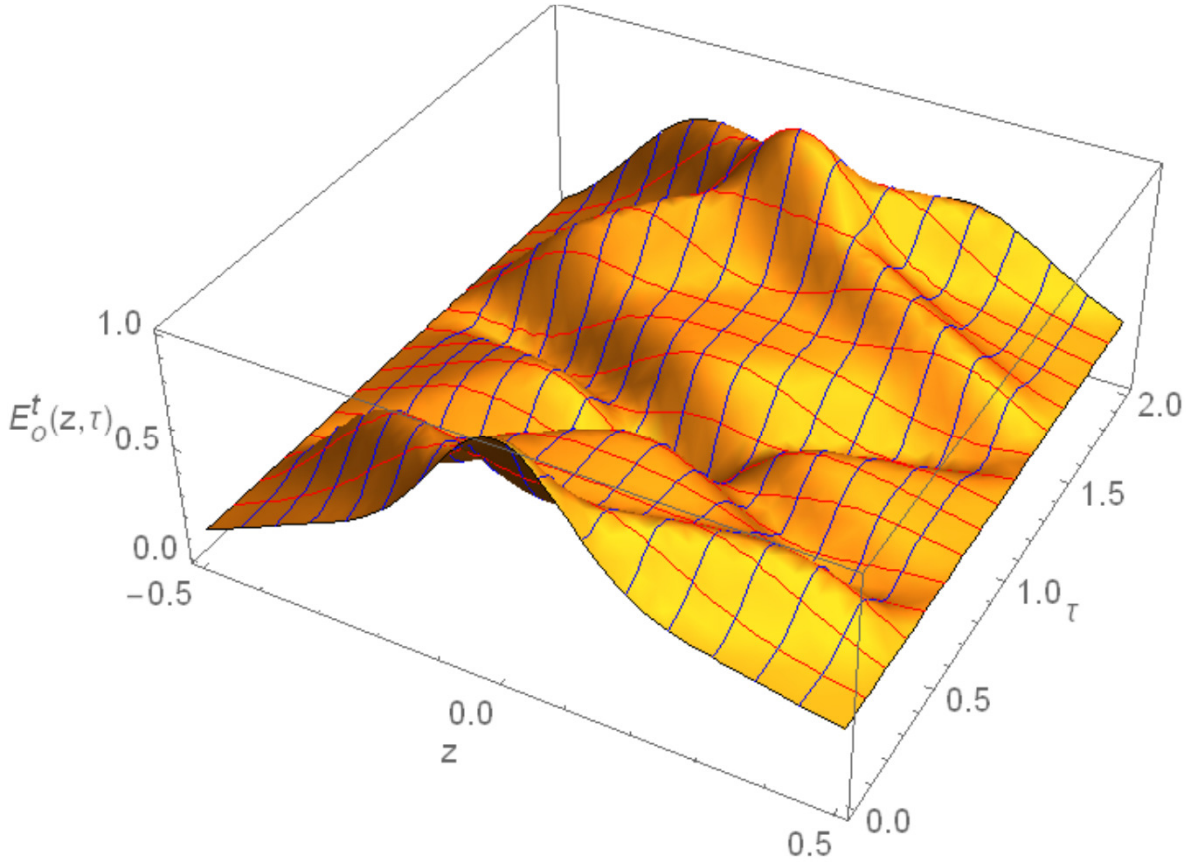
$$u(z,\bar{\tau}) = z + a\gamma(z)\bar{\tau}$$

$$\gamma(z) = \tfrac{1}{2}\left(1 + e^{-\gamma_e \hat{z}^2}\right)$$

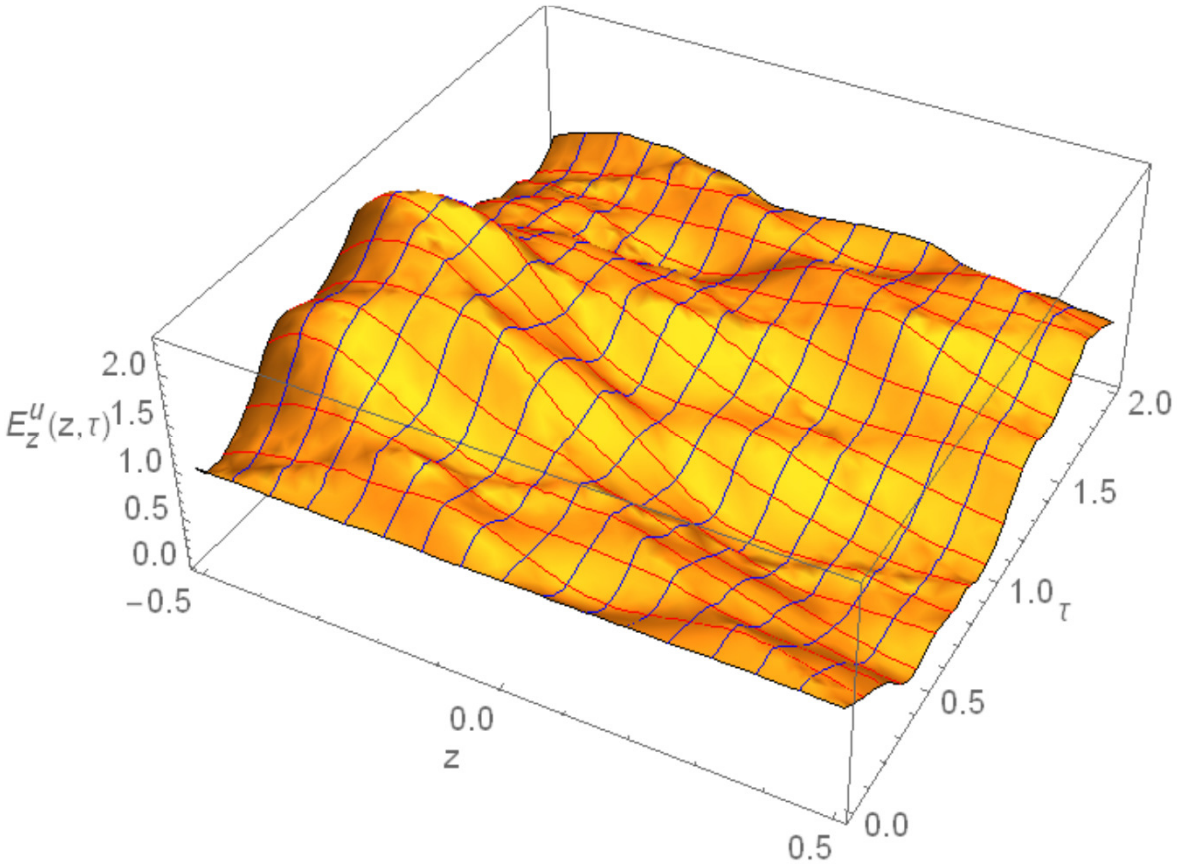


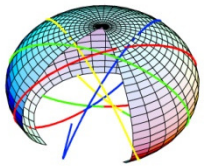


Frame transformation—time

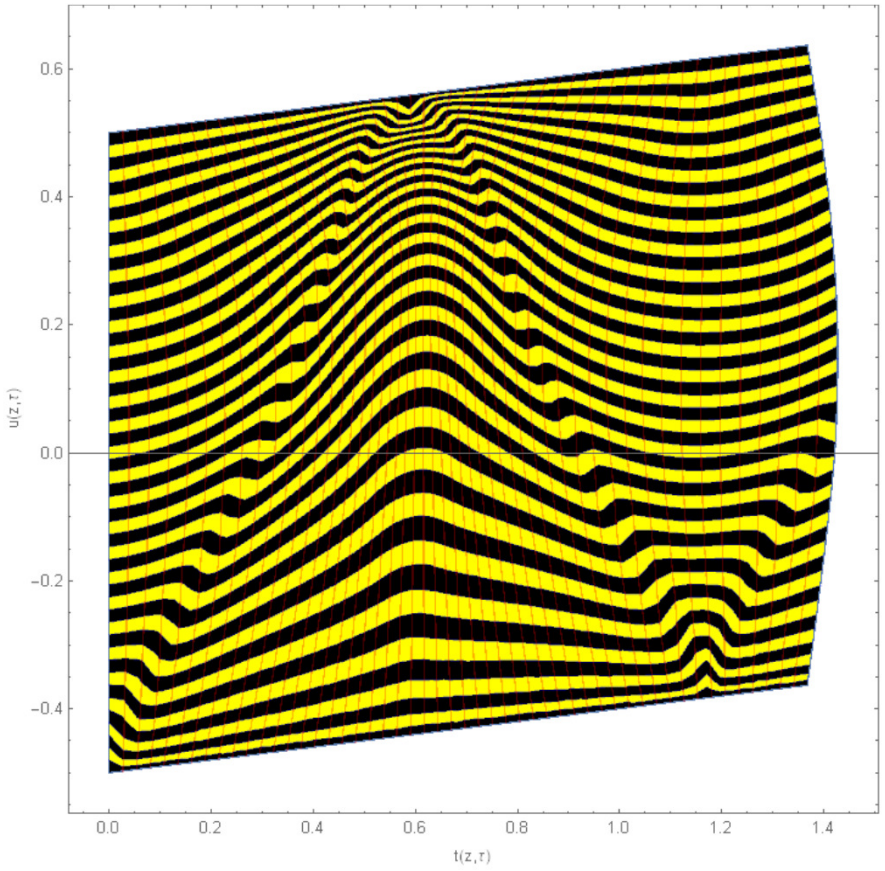


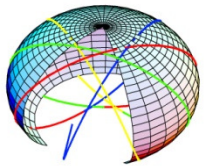
Frame transformation--position



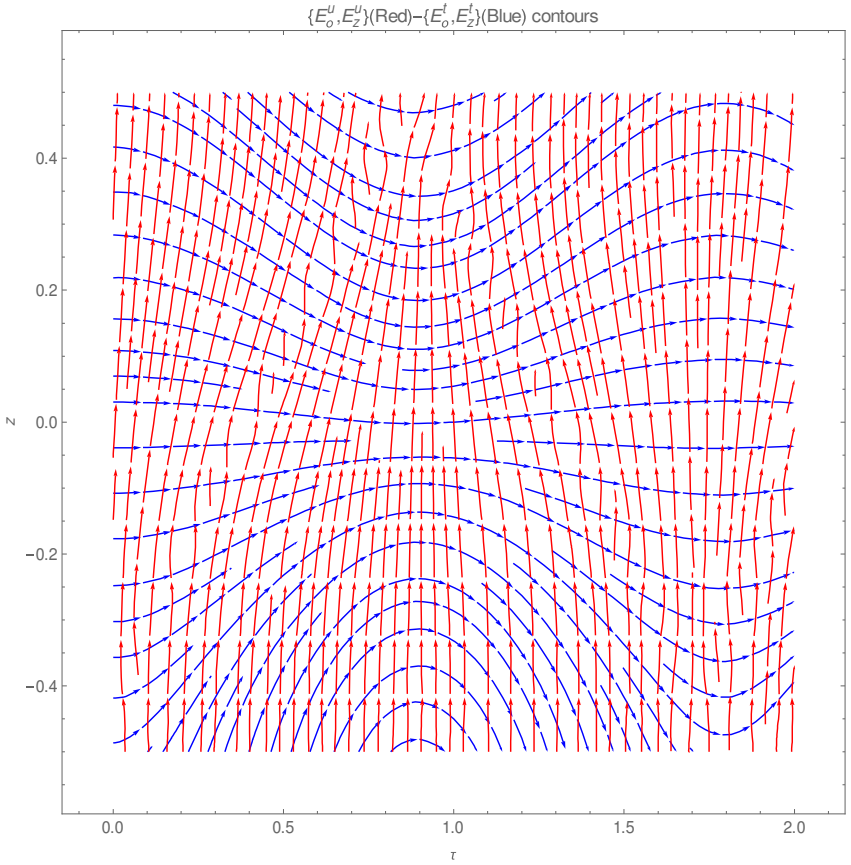


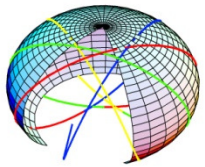
Position versus time



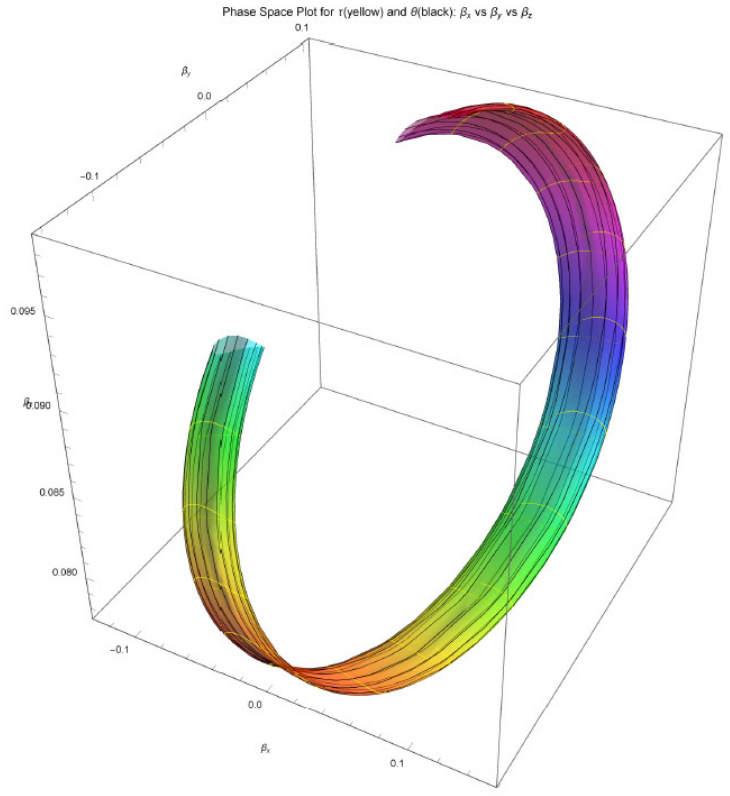


Detailed frame transformation

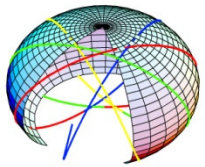




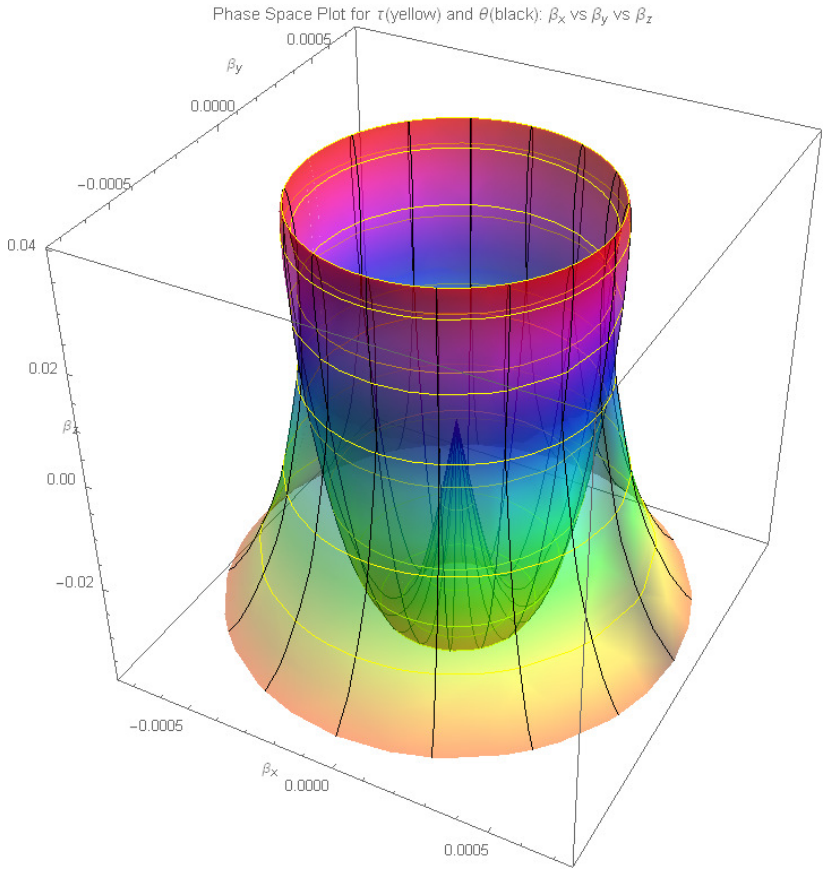
Beta from WTC 2014

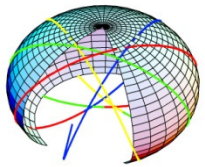


z = −0.20, r = 1 / 5



Beta from 3+1 model





Conclusions

- We are helped to visualize "time" and "frames"
  - Our Harmonic Wave equation is linear but the "DC" equations are not as functions of physical time
  - We have DC, AC steady State, AC Transient and AC harmonic solutions
- Just as in in circuits, the AC behaviors exhibit
  - Possible morphogenetic behaviors (Thom)
  - Breaking waves
  - Possible chaotic behaviors
    - Space filling curves
- As engineering, these behaviors reflect the global!