Linear Recursion Method

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1.1 Introduction

In Vol. 2 (Thomas, 2017) we outlined a strategy for computing results for decision process theory models. This is expanded on in the white paper on this site[1]. The starting point is the idea that meaningful results from the theory on decisions can be obtained by considering steady state phenomena, not unlike the idea of studying AC circuits in electrical engineering. The resultant equations are non-linear elliptic partial differential equations. The problem is that such equations currently don't admit a simple numerical solution strategy.

In Vol. 2 we chose the method of lines used in Mathematica for solving non-linear equations. The advantage was that it dealt with the non-linear nature of the equations with imposed periodic boundary conditions. In Mathematica, the method is well developed and fairly fast, which is important given the number of coupled equations involved. However, we noted that near the boundaries, the solutions appeared unstable, as we might expect since strictly speaking, the method does not apply. We suggest here a different path, based in part on suggestions from (Hawking & Ellis, 1973). We suggest that the approach is to

[1] http://decisionprocesstheory.com/white-papers/vol-2-resources/stationary-ownership-model/

approximate the non-linear PDE with a linear PDE and recursively solve these equations: the ***linear recursion method***. If the series converges we have a way to solve the equations. By choosing this method, we assert that the overall strategy of Vol. 2 still applies. It opens up the possibility of solving a wider class of models if we adopt a covariant gauge condition on the equations in a holonomic frame. However, we anticipate that this method may be slower and so the challenge may be to make the approach more efficient.

In this white paper, we explore the consequences of this approach by starting with a toy model, the non-linear Sine-Gordon equation (Sec. 1.2). We apply the method of lines and then the recursive approach. We find that the recursive approach converges, though we can't rule out the possibility of bifurcations. Based on our experience here, there is nothing to prevent us from applying this method to the numerical calculations in Vol. 2.

In Sec. 1.3 we show how the covariant gauge condition is dealt with for elliptic equations using Dirichlet boundary conditions. Then in Sec. 1.4 we show that in a covariant gauge and a holonomic frame, the field equations for decision process theory can be reduced to a recursive set of linear elliptic partial differential equations (Hawking & Ellis, 1973). These equations can then be expressed in terms of active and inactive coordinates (Sec. 1.5).

1.2 Sine Gordon Equation

The Sine-Gordon equation is a non-linear equation, which in two dimensions is:

$$-\nabla^2 u(x,y) = \lambda \sin u(x,y) \tag{1.1}$$

It is clearly non-linear in $u(x,y)$. The idea is to replace this equation with the ***linear recursive method***, which is a series of equations that are linear:

$$
\begin{aligned}
&u_0 = k\\
&\nabla^2\delta u_1 + \lambda\delta u_1 \cos u_0 = -\nabla^2 u_0 - \lambda \sin u_0\\
&\nabla^2\delta u_2 + \lambda\delta u_2 \cos u_1 = \lambda\left(\left(u_1 - u_0\right)\cos u_0 - \sin u_1 + \sin u0\right)\\
&\cdots\\
&\nabla^2\delta u_n + \lambda\delta u_n \cos u_{n-1} = \lambda\left(\left(u_{n-1} - u_{n-2}\right)\cos u_{n-2} - \sin u_{n-1} + \sin u_{n-2}\right)\\
&\delta u_n = u_n - u_{n-1}\\
&u\left(x,y\right) = \lim_{n\to\infty} u_n\left(x,y\right)
\end{aligned}
\tag{1.2}
$$

Each equation is a linear elliptic partial differential equation. The Dirichlet boundary condition for δu_n is zero, not matter what the given Dirichlet boundary condition is for u (and hence each of the u_n). We have removed the Laplacian operator on the right-hand-side using the previous iteration solutions. This approach will be echoed again when we deal with the Einstein equations.

There are two approaches to take: 1) we can solve Eq. (1.1) using the ***method of lines*** or 2) we can use the *linear recursive method* and solve each of the equations in the series, Eq. (1.2) to obtain the limit. For this simple example, we do both and compare the results in a Mathematica notebook on this site[2]. First, the method of lines gives Fig. 1.1. We see some evidence of instability around the boundary edges. We also note that we were forced to use periodic boundary conditions in order to obtain a solution. For the general case, we remove the requirement that we use periodic boundary conditions (though we can still impose them if we wish).

[2] http://decisionprocesstheory.com/wp-content/uploads/2016/10/Linear-Recursion-Method.cdf

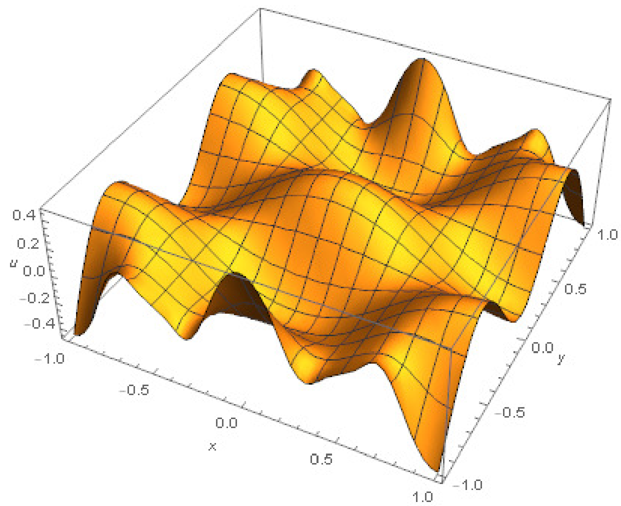

Figure 1.1. Method of Lines for $\lambda = 40,\ scale = 0.3$

Second, we compute the solution based on a series of iterations. The details of this and the previous result are provided in a Mathematica notebook. Typical results are shown in Fig. 1.2 for periodic boundary conditions for x and Dirichlet boundary conditions for y. The idea here was to match Fig. 1.1 as closely as possible.

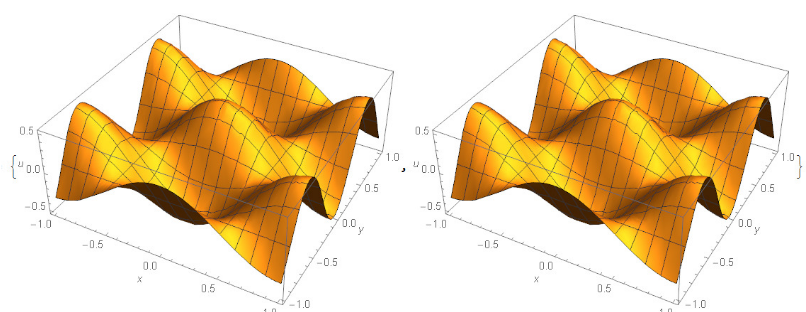

Figure 1.2. *Linear Recursive Method*, showing the last 2 iterations of a total of 5, showing that they have converged. The value $\lambda = 40$ and the value on the boundary is $\frac{1}{3}\cos \pi x$.

There are several noteworthy observations. We have chosen a coupling λ that is large. The solutions are much more behaved around the boundaries than the method of lines. The solutions are approximately the same in the middle, though this is not guaranteed. It is the case for the example we chose here. We believe that these solutions are instructive. The iterative solution appears to work even for large couplings.

It is also significant that the shapes of the solutions obtained are not simple extrapolations of the boundary value behaviors input for larger couplings. Though the shapes around the origin may be similar, the boundary behaviors are different. Moreover, the value at the origin reflects the non-linear behavior of the equation and doesn't scale with the values on the boundaries. This global behavior is thus quite distinct from the local behaviors, showing the power of using differential geometry models to investigate global properties. We see this in more detail if we study a pure Dirichlet solution that is constant on the boundary.

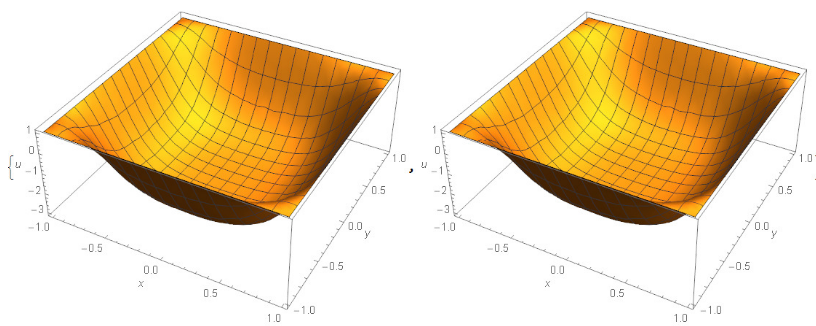

Figure 1.3. Pure Dirichlet boundary condition $u=1$, with $\lambda=40$ and $n=15$ iterations.

We conclude that it is preferable to set the values on the boundary not in the center because on the boundary we are more likely to be able to identify "known conditions". On the boundary, we may be far enough away from non-linear behaviors to accurately assess these values. In some ways, the boundary is where we might see local behaviors without acceleration effects. The significant value of the theory is to allow us to extrapolate from the boundary to the interior.

Although we make the argument for the steady state case, the results also apply more generally. Even for time dependent solutions that would show transient effects, there will still be a subset of the equations that are constraints on an initial surface. The remarks apply also to that initial surface.

1.3 Covariant Gauge Condition Example

In holonomic frames, the gauge conditions are often expressed in covariant form. In this form the equations are Laplace like equations and manifestly elliptic partial differential equations, albeit non-linear ones. A difficulty with the gauge conditions is that they are imposed on the boundary and then shown to hold everywhere inside the boundary based on the covariant elliptic partial differential equations. We need to see exactly what boundary conditions are imposed, since they are not pure Dirichlet conditions. We want to see that the equations once imposed then automatically hold on the inside of the boundary. We do that in this section based on a simple toy model.

We wish to deal with the covariant gauge for elliptic equations using Mathematica. We need to have a linear elliptic equation, though the coefficients may be complicated functions. We apply the Dirichlet condition on the surface of each boundary and the Neumann condition on the normal. Here is an example in two dimensions (conductor):

$$\begin{aligned}
&\Omega = \text{Rectangle}[\{-1,-1\},\{1,1\}] \\
&\nabla^2 A_x = j_x = x \\
&\nabla^2 A_y = j_y = -y \\
&\nabla \bullet A = 0 \\
&A_y(\pm 1, y) = 0,\ A_x(x, \pm 1) = 0 \\
&\partial_x A_x(\pm 1, y) = 0,\ \partial_y A_y(x, \pm 1) = 0
\end{aligned} \tag{1.3}$$

We use Mathematica with the above arbitrary choice for a conserved current. The results are computed in a Mathematica notebook[3] and show that the existing Mathematica tools are able to solve the equations, Fig. 1.4.

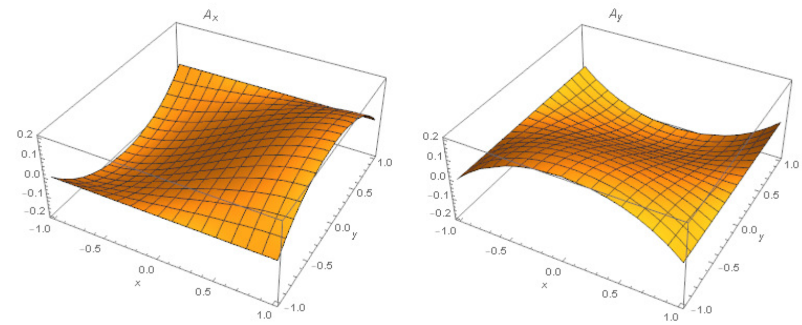

Figure 1.4. Two-dimensional vector potentials using Mathematica

Of interest is the gauge condition itself in the interior of the box, which is shown in Fig. 1.5. These results along with corresponding results are provided in the Mathematica notebook.

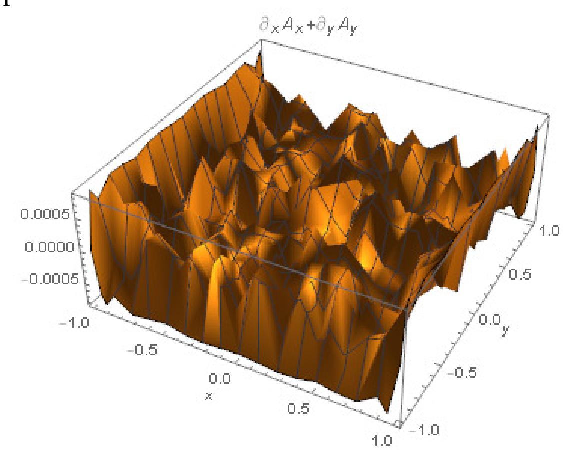

Figure 1.5. Gauge condition inside the boundary

[3] http://decisionprocesstheory.com/wp-content/uploads/2016/10/Covariant-Gauge-Condition-Examples-in-2D-and-3D.cdf

We see that the gauge condition is effectively zero inside the boundary to fairly high precision. This suggests how we can treat covariant gauges in general, including those that occur in decision process theory where the equations are much more complicated. The only caveat is that the elliptic partial differential equations remain linear.

1.4 Holonomic Frame Equations

In Vol. 2, we looked at a specialized set of models that could be computed in the orthonormal coordinate frame. We applied the *method of lines* to obtain numerical solutions. We can of course return to those models and apply instead the *linear recursion method*. We hope to do that at some point. However we think a more interesting approach is to expand that set of models to an arbitrary set of models in which the behavior is stationary. In both approaches, we think we need to use the covariant gauge described in the previous section. In both cases we need to linearize the equations. It might be fairly straightforward for the models in Vol. 2. For the more general case, we have to return to the general formulation in Vol. 1.

Decision process theory results in a set of least path equations, which for steady state take the form of elliptic non-linear partial differential equations. The goal is to linearize the equations in such a way that the exact solution can be found by a recursive iteration. We start in a holonomic frame and assume that there is a background metric that is known. We linearize the equations, replacing the unknown metric everywhere with this background metric in such a way that the equation is linear in the unknown metric. We solve the resultant equations, which for steady state, will be elliptic partial differential equations. The solution will be the new background metric.

We recall a few of the key steps that involved in obtaining the least action equations, which are first order derivatives of the connection. The metric g_{ab} determines the connection, which since the frame is holonomic, can be written in terms of first order derivatives of the metric.

$$\varpi_{abc} = \tfrac{1}{2}\left(g_{bd}g_{ce}\partial_a g^{de} - g_{ae}g_{cd}\partial_b g^{de} - g_{ad}g_{be}\partial_c g^{de}\right) \tag{1.4}$$

The resultant equations will then be second order derivatives in the metric.

Let $\tilde{g}_{ab}$ be the background metric and let $X^a{}_{|b}$ represent the covariant derivative with respect to that metric; we represent the covariant derivative with respect to the unknown metric be $X^a{}_{;b}$. According to (Hawking & Ellis, 1973) the difference between the connection and background connection transforms as a tensor and is:

$$\begin{aligned}
\delta\varpi^a_{bc} &\equiv \varpi^a_{bc} - \tilde{\varpi}^a_{bc}\\
\delta\varpi^a_{bc} &= \tfrac{1}{2}\left(2\varpi^a_{bc} - \tilde{\varpi}^a_{bc} - \tilde{\varpi}^a_{cb}\right)\\
\delta\varpi^a_{bc} &= \tfrac{1}{2}\left(\partial_r g^{pq} + \tilde{\varpi}^p_{sr}g^{sq} + \tilde{\varpi}^q_{sr}g^{ps}\right)\left(g_{bp}g_{cq}g^{ar} - g_{bp}\delta^r_c\delta^a_q - g_{cp}\delta^r_b\delta^a_q\right)\\
\delta\varpi^a_{bc} &= \tfrac{1}{2}g^{ar}\left(-g_{cb|r} + g_{br|c} + g_{rc|b}\right)\\
\delta\varpi^a_{bc} &= \tfrac{1}{2}g^{pq}_{|r}\left(g_{bp}g_{cq}g^{ar} - g_{bp}\delta^r_c\delta^a_q - g_{cp}\delta^r_b\delta^a_q\right)
\end{aligned} \tag{1.5}$$

Since the covariant derivative of the background metric, with respect to that metric, is zero, we can replace the first factor with the covariant derivative of the difference $\delta g_{ab} \equiv g_{ab} - \tilde{g}_{ab}$:

$$\delta\varpi^a_{bc} = \tfrac{1}{2}\delta g^{pq}_{|r}\left(g_{bp}g_{cq}g^{ar} - g_{bp}\delta^r_c\delta^a_q - g_{cp}\delta^r_b\delta^a_q\right) \tag{1.6}$$

The next step is to consider the difference between the curvature tensor with the exact metric and the curvature tensor based on the background metric. With some effort the following result follows:

$$\begin{aligned}
-R_{ab} &= \partial_d\varpi^d{}_{ab} - \partial_b\varpi^d{}_{da} + \varpi^d{}_{ab}\varpi^e{}_{de} - \varpi^d{}_{ea}\varpi^e{}_{bd}\\
-\tilde{R}_{ab} &= \partial_d\tilde{\varpi}^d{}_{ab} - \partial_b\tilde{\varpi}^d{}_{da} + \tilde{\varpi}^d{}_{ab}\tilde{\varpi}^e{}_{de} - \tilde{\varpi}^d{}_{ea}\tilde{\varpi}^e{}_{bd}\\
\delta R_{cd} &= -\delta\varpi^e{}_{cd|e} + \delta\varpi^e{}_{ce|d} - \delta\varpi^f{}_{cd}\delta\varpi^e{}_{fe} + \delta\varpi^f{}_{ce}\delta\varpi^e{}_{df}
\end{aligned} \tag{1.7}$$

We see the possibility of simplifying this expression if we keep only first order terms. Before doing that however, we write the field equations in terms of the differences defined above.

The field equations depend on $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab}$, so the relevant quantity is the difference of this tensor relative to the two metrics:

$$\delta G^{ab} = R^{ab} - \tfrac{1}{2} g^{ab} R - \left(\tilde{R}^{ab} - \tfrac{1}{2} \tilde{g}^{ab} \tilde{R} \right) \tag{1.8}$$

This expression is certainly complicated, but it contains the following terms:

$$\begin{aligned} U^{ab} \equiv\ & \tfrac{1}{2} g^{cd} \delta g^{ab}{}_{|cd} - \tfrac{1}{2} g^{ca} \psi^{b}{}_{c} - \tfrac{1}{2} g^{cb} \psi^{a}{}_{c} \\ & + \tfrac{1}{2} g^{ab} \left(\psi^{c}{}_{c} - \tfrac{1}{2} g_{cd} g^{ef} \delta g^{cd}{}_{|ef} \right) \\ \psi^{b}{}_{c} \equiv\ & \delta g^{bf}{}_{|fc} - \tfrac{1}{2} g^{bf} g_{de} \delta g^{de}{}_{|fc} \end{aligned} \tag{1.9}$$

We can expand these expression to first order in the difference δg_{ab}, which we indicate by $\doteq$:

$$\begin{aligned} U^{ab} \doteq\ & \tfrac{1}{2} \tilde{g}^{cd} \delta g^{ab}{}_{|cd} - \tfrac{1}{2} \tilde{g}^{ca} \psi^{b}{}_{c} - \tfrac{1}{2} \tilde{g}^{cb} \psi^{a}{}_{c} \\ & + \tfrac{1}{2} \tilde{g}^{ab} \left(\psi^{c}{}_{c} - \tfrac{1}{2} \tilde{g}_{cd} \tilde{g}^{ef} \delta g^{cd}{}_{|ef} \right) \\ \phi^{bc}{}_{d} \equiv\ & \delta g^{bc}{}_{|d} - \tfrac{1}{2} \tilde{g}^{bc} \tilde{g}_{ef} \delta g^{ef}{}_{|d} \\ \psi^{b}{}_{c} \doteq\ & \left(\delta g^{bd}{}_{|d} - \tfrac{1}{2} \tilde{g}^{bd} \tilde{g}_{ef} \delta g^{ef}{}_{|d} \right)_{|c} = \phi^{bd}{}_{d|c} \end{aligned} \tag{1.10}$$

We identify the gauge condition as:

$$\phi^{bc}{}_{c} = \delta g^{bc}{}_{|c} - \tfrac{1}{2} \tilde{g}^{bc} \tilde{g}_{ef} \delta g^{ef}{}_{|c} = 0 \tag{1.11}$$

This condition is first order in the unknown metric differences and also a first order differential equation in the background covariant derivative. The interesting property is that when the gauge condition is satisfied, the function U^{ab} reduces to a form that is a Laplacian form since $\psi^{a}{}_{b} \doteq 0$:

$$U^{ab} \doteq \tfrac{1}{2} \tilde{g}^{cd} \left(\delta g^{ab} - \tfrac{1}{2} \tilde{g}^{ab} \tilde{g}_{ef} \delta g^{ef} \right)_{|cd} \tag{1.12}$$

This is helpful because we can compute $W^{ab} = \delta G^{ab} + U^{ab}$ and show that it has no first or second order covariant derivatives:

$$W^{ab} \doteq \left(\tilde{g}^{ap}\tilde{g}^{bq}\tilde{R}_{pcdq} - \tfrac{1}{2}\tilde{g}^{ab}\tilde{R}_{cd}\right)\delta g^{cd} + \tfrac{1}{2}\left(\tilde{R}_{cd} - \tfrac{1}{2}\tilde{g}_{cd}\tilde{R}\right)\left(\delta g^{ca}\tilde{g}^{bd} + \delta g^{cb}\tilde{g}^{ad}\right) \tag{1.13}$$

This is in fact linear in the unknown metric difference. The coefficients depend only on the background metric and its first and second order derivatives.

We are thus in a position to write the general equation:

$$\tfrac{1}{2}\tilde{g}^{ef}\left(\delta g^{ab}{}_{\backslash ef} - \tfrac{1}{2}\tilde{g}^{ab}\tilde{g}_{pq}\delta g^{pq}{}_{\backslash ef}\right) \doteq W^{ab} + \kappa T^{ab} + \tilde{R}^{ab} - \tfrac{1}{2}\tilde{g}^{ab}\tilde{R} \tag{1.14}$$

The left-hand side is an elliptic partial differential operator on the metric difference; the right-hand side is presumed known in terms of the background metric. There will be additional changes on the right once we add the concept of stationary ownership and introduce an unknown energy flow V^a along with scalar parameters for the energy density and pressures. Nevertheless, we retain the posture (Hawking & Ellis, 1973) that these equations are known and soluble in principle using Mathematica tools. We will address these issues at a later time.

As with the Sine-Gordon example, we start with an arbitrary choice for the first background metric (we can take it to be Minkowski metric for example). We then compute the new metric from the above equations. We keep iterating until we detect convergence. This is the part that may be numerically challenging.

1.5 Active and Inactive Dimensions

In principle the equations from the previous section apply to our field equations with or without a split between active and inactive coordinates. The equations that result are tensors and therefore can be transformed into any frame, including those frames in which the coordinates are not holonomic. There may be some advantage to go to such frames: with inactive components, it is easier to identify the gauge invariant quantities using the frame from Vol. 2 in which the active components remain holonomic but are orthogonal to the inactive components that are not. In that frame, the decision process theory payoffs are explicitly separated.

In this section we collect from Vol. 2 some of the equations that are needed for that separation. The key idea is that there is a frame rotation in going from the background gauge to the gauge invariant frame. Before looking at the components of the curvature tensor, we need to make the comparisons relative to the background gauge.

The rotation moves the inactive space into the active space so we expect that the mixed metric component in the rotated space will have a form such as $\breve{g}_{aj} = -\gamma_{jk}\delta A^{k}{}_{a}$, proportional to the rotation. We use the hook to represent the exact metric evaluated in the background frame:

$$\begin{aligned} U^{a} &= \tilde{U}^{a} = \breve{U}^{a} = dx^{a} \\ \breve{U}^{j} &= \tilde{U}^{j} = d\xi^{j} + \tilde{A}^{j}{}_{a} dx^{a} \\ U^{j} &= d\xi^{j} + A^{j}{}_{a} dx^{a} = \breve{U}^{j} + \delta A^{j}{}_{a} \breve{U}^{a} \end{aligned} \tag{1.15}$$

The background metric uses the coordinates $\{\breve{U}^{a} \;\; \breve{U}^{j}\}$ whereas the exact metric uses the coordinates $\{U^{a} \;\; U^{j}\}$. Because the inactive frame is not holonomic, the two frames don't align. From Eq. (1.15), we have the frame transformations to and from the background frame:

$$\begin{aligned} \breve{\mathbf{U}} &= \left(\breve{U}^{\alpha}{}_{\beta}\right) = \begin{pmatrix} \breve{U}^{j}{}_{k} & \breve{U}^{j}{}_{b} \\ \breve{U}^{a}{}_{k} & \breve{U}^{a}{}_{b} \end{pmatrix} = \begin{pmatrix} \delta^{j}_{k} & \delta A^{j}_{b} \\ 0 & \delta^{a}_{b} \end{pmatrix} \\ \breve{\mathbf{U}}^{-1} &= \left(\breve{U}^{-1\beta}{}_{\alpha}\right) = \begin{pmatrix} \breve{U}^{-1k}{}_{j} & \breve{U}^{-1k}{}_{b} \\ \breve{U}^{-1a}{}_{j} & \breve{U}^{-1a}{}_{b} \end{pmatrix} = \begin{pmatrix} \delta^{k}_{j} & -\delta A^{k}_{a} \\ 0 & \delta^{b}_{a} \end{pmatrix} \end{aligned} \tag{1.16}$$

As examples, here are transformations that will prove useful:

$$
\begin{aligned}
&\breve{X}_a = X_\beta \breve{U}^{-1\beta}{}_a = X_a - X_j \delta A_a^j \\
&\breve{X}_j = X_\beta \breve{U}^{-1\beta}{}_j = X_j \\
&\breve{X}^j = X^j + \delta A_a^j X^a \\
&\breve{X}^a = X^a \\
&\breve{\gamma}_{jk} = \gamma_{jk} \\
&\breve{g}_{ja} = -\gamma_{jk} \delta A_a^k \\
&\breve{g}_{ab} = g_{ab} + \gamma_{jk} \delta A_a^j \delta A_b^k \\
&\breve{\gamma}^{jk} = \gamma^{jk} + g^{ab} \delta A_a^j \delta A_b^k \\
&\breve{g}^{aj} = \delta A_b^j g^{ab}
\end{aligned}
\tag{1.17}
$$

We have enough information to transform any tensor as well as computing the orientation potentials in the rotated frame. The inactive metric doesn't change and the active metric change is second order; we suspect that we can ignore that. The mixed tensor component is new, is first order and shows that the effect of the rotation is to destroy the orthogonality of the active and inactive spaces.

According to (Hawking & Ellis, 1973), the difference between two connections ***in the same frame*** is a tensor. Thus in any frame, the exact solution will be compared to the background gauge solution in the reference frame of the background gauge. We use vol. 2 to compute the exact connection with axes oriented for the background gauge, starting with:

$$
\begin{aligned}
&d\breve{\mathbf{U}}^j = \breve{\mathbf{F}}^j = \tfrac{1}{2}\breve{C}^j{}_{ab}\breve{\mathbf{U}}^a \wedge \breve{\mathbf{U}}^b = \tilde{\mathbf{F}}^j = \tfrac{1}{2}\tilde{C}^j{}_{ab}\breve{\mathbf{U}}^a \wedge \breve{\mathbf{U}}^b \\
&\breve{C}^\alpha{}_{\beta\gamma} = \tilde{C}^\alpha{}_{\beta\gamma} \Rightarrow \breve{C}_{\alpha\beta\gamma} = \breve{\gamma}_{\alpha\delta}\tilde{C}^\delta{}_{\beta\gamma} \\
&\breve{\Gamma}_{\alpha\beta\gamma} \equiv \tfrac{1}{2}\left(-\mathbf{\Delta}_\alpha\breve{\gamma}_{\beta\gamma} + \mathbf{\Delta}_\beta\breve{\gamma}_{\gamma\alpha} + \mathbf{\Delta}_\gamma\breve{\gamma}_{\alpha\beta}\right) = \breve{\gamma}_{\alpha\delta}\breve{\Gamma}^\delta{}_{\beta\gamma} \\
&\breve{\varpi}_{\alpha\beta\gamma} = \tfrac{1}{2}\left(\breve{C}_{\alpha\beta\gamma} + \breve{C}_{\beta\gamma\alpha} - \breve{C}_{\gamma\alpha\beta}\right) + \tfrac{1}{2}\left(-\mathbf{\Delta}_\alpha\breve{\gamma}_{\beta\gamma} + \mathbf{\Delta}_\beta\breve{\gamma}_{\gamma\alpha} + \mathbf{\Delta}_\gamma\breve{\gamma}_{\alpha\beta}\right)
\end{aligned}
\tag{1.18}
$$

Now in fact we won't need the explicit expression for these elements. To first order, we need only the expressions for the metric in the rotated frame, Eq. (1.17). The difference $\delta\varpi_{\alpha\beta\gamma} = \breve{\varpi}_{\alpha\beta\gamma} - \tilde{\varpi}_{\alpha\beta\gamma}$ transforms like a tensor and allows us to carry over the results from the last section.

$$\begin{aligned}
&\phi^{\alpha\beta}{}_{\beta} = \delta g^{\alpha\beta}{}_{|\beta} - \tfrac{1}{2}\,\tilde{g}^{\alpha\beta}\,\tilde{g}_{\gamma\delta}\delta g^{\gamma\delta}{}_{|\beta} = 0 \\
&W^{\alpha\beta} \doteq \left(\tilde{g}^{\alpha\varepsilon}\,\tilde{g}^{\beta\eta}\tilde{R}_{\varepsilon\gamma\delta\eta} - \tfrac{1}{2}\,\tilde{g}^{\alpha\beta}\tilde{R}_{\gamma\delta}\right)\delta g^{\gamma\delta} \\
&\qquad + \tfrac{1}{2}\left(\tilde{R}_{\gamma\delta} - \tfrac{1}{2}\,\tilde{g}_{\gamma\delta}\tilde{R}\right)\left(\delta g^{\alpha\gamma}\,\tilde{g}^{\beta\delta} + \delta g^{\beta\gamma}\,\tilde{g}^{\alpha\delta}\right) \\
&\tfrac{1}{2}\,\tilde{g}^{\gamma\delta}\left(\delta g^{\alpha\beta}{}_{|\gamma\delta} - \tfrac{1}{2}\,\tilde{g}^{\alpha\beta}\,\tilde{g}_{\varepsilon\eta}\delta g^{\varepsilon\eta}{}_{|\gamma\delta}\right) \doteq W^{\alpha\beta} + \kappa T^{\alpha\beta} \\
&\qquad\qquad + \tilde{R}^{\alpha\beta} - \tfrac{1}{2}\,\tilde{g}^{\alpha\beta}\tilde{R}
\end{aligned} \tag{1.19}$$

We have the gauge condition, the linear term and the form of the elliptic equation.

Strictly speaking the metric difference is between the background metric and the rotated metric. But to this order, what we have is:

$$\begin{aligned}
\delta g^{ab} &= \breve{g}^{ab} - \tilde{g}^{ab} = g^{ab} - \tilde{g}^{ab} \\
\delta\gamma^{jk} &= \breve{\gamma}^{jk} - \tilde{\gamma}^{jk} \doteq \gamma^{jk} - \tilde{\gamma}^{jk} \\
\delta g^{ak} &= \breve{g}^{ak} - 0 = g^{ab}\delta A^{k}_{b} \doteq \tilde{g}^{ab}\delta A^{k}_{b}
\end{aligned} \tag{1.20}$$

We are thus computing the changes of interest, including the mixing term set by the payoffs. As we iterate through these recursive equations, the differences should get smaller and smaller after a sufficiently number of iterations. If that happens, we then get a solution to the non-linear equations. This particular frame of reference thus works the same way, including dealing with the vector potentials.

In addition to Eq. (1.19) we will need to compute the second order elliptic operator as well as the curvature components with the background metric. We will need the connection terms for the elliptic operator Vol. 2:

$$
\begin{aligned}
\tilde{\varpi}_{ijk} &= 0 \\
\tilde{\varpi}_{ajk} &= -\tfrac{1}{2}\partial_a \tilde{\gamma}_{jk} \\
\tilde{\varpi}_{jak} &= \tfrac{1}{2}\partial_a \tilde{\gamma}_{kj} \\
\tilde{\varpi}_{jka} &= \tfrac{1}{2}\partial_a \tilde{\gamma}_{jk} \\
\tilde{\varpi}_{jab} &= \tfrac{1}{2}\tilde{\gamma}_{jk}\tilde{F}^k{}_{ab} \\
\tilde{\varpi}_{ajb} &= -\tfrac{1}{2}\tilde{\gamma}_{jk}\tilde{F}^k{}_{ab} \\
\tilde{\varpi}_{abj} &= -\tfrac{1}{2}\tilde{\gamma}_{jk}\tilde{F}^k{}_{ab} \\
\tilde{\varpi}_{abc} &= \tfrac{1}{2}\left(-\partial_a \tilde{g}_{bc} + \partial_b \tilde{g}_{ca} + \partial_c \tilde{g}_{ab}\right)
\end{aligned}
\tag{1.21}
$$

The curvature components can be obtained from Ch. 4, Exs. 6-8, Vol. 2.

As a first application of these ideas, we quote the results of the calculation for the gauge conditions. There are two: one for the active components and one for the mixed components. The active component gauge conditions are:

$$
\begin{pmatrix}
\dfrac{1}{\sqrt{|\tilde{g}\tilde{\gamma}|}}\partial_b\left(\sqrt{|\tilde{g}\tilde{\gamma}|}\tilde{g}_{ac}\delta g^{cb}\right) - \tfrac{1}{2}\partial_a\left(\tilde{g}_{cd}\delta g^{cd} + \tilde{\gamma}_{jk}\delta\gamma^{jk}\right) \\
-\tfrac{1}{2}\partial_a \tilde{g}_{cd}\delta g^{cd} - \tilde{\gamma}_{jk}\tilde{F}^k{}_{ab}\tilde{g}^{bc}\delta A^j{}_c
\end{pmatrix} \doteq 0 . \tag{1.22}
$$

The mixed gauge conditions are:

$$
\partial_b\left(\sqrt{|\tilde{g}\tilde{\gamma}|}\tilde{\gamma}_{ij}\tilde{g}^{ba}\delta A^j{}_a\right) \doteq 0 . \tag{1.23}
$$

The latter agrees with Vol. 1, whereas the former differs in a few details, probably slight errors in the earlier calculation.

1.6 Conclusions

We have shown that it is plausible to solve the partial differential equations of decision process theory using the *linear recursion method* and a linearized form of the covariant gauge condition. We have suggested that the gauge imposed on the boundary as Dirichlet and Neumann conditions will be maintained. Implicitly we have argued that focusing on the boundaries might be a powerful way to look at decision process, just as it is a powerful way to look at electrical engineering problems. The non-linear character of the problem may not manifest itself as strongly on the boundary if the boundaries are carefully chosen. Clearly the next challenge is to support this point of view.