Stationary Ownership Model

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1.1 Introduction

The goal is to get a form for the energy momentum tensor articulated in the ***stationary holonomic frame***. This is a holonomic frame in which the active strategies commute and are ***holonomic***": a holonomic coordinate is one specified by a scalar field, not unlike a potential, where constant surfaces define each coordinate value. Since the behaviors are stationary, time is inactive and is assumed to mutually commute with the inactive strategies. The inactive strategies need not be treated as holonomic as discussed in the next section. We may choose to treat time as holonomic along with the active strategies. The arguments below require that we be able to transform to a frame in which at that point, the flow vanishes. We call that the ***strictly stationary frame***.

In focusing on stationary behaviors, we take our cue from engineering in which they provide a useful guide to complex behaviors. We have in mind AC circuits as a natural way to proceed from DC circuits. Nevertheless, transient behaviors, which we ignore, are important and this approach provides useful insights.

1.2 The Stationary Holonomic Frame

The ***stationary holonomic frame*** is obtained for models in which the inactive player strategies mutually commute. We can choose this frame to be a holonomic basis for the active strategies (and time) with metric g_{ab} and a non-holonomic basis for the inactive dimensions that lead to the metric components γ_{jk} along with the associated payoffs $F^{j}{}_{ab}$. This decomposition is dealt with in detail in Vol. 2.

Our goal is to create in this frame an ownership structure that is

- A projection operator
- Is orthogonal to the flow
- Depends on only the above components along with the player engagements V_j and the flow components V_υ.
- The projection operator is defined in the ***strictly stationary frame*** where the player flows vanish. In this frame we believe that ownership will have the desired meaning: it is isotropic and projects out just the active and inactive strategies that are owned.
- The operator should be a diagonal matrix with unit elements for each strategy that is in the ownership group for that player class $o[J]$.

1.3 The Transformation U to No Flows

The first step is to linearly transform to a frame in which the contravariant and covariant flows are zero. We do this in two steps. First we note that there is always a frame in which the time components are orthogonal to the strategies:

$$\begin{aligned}
&\bar{g}_{o\upsilon} = 0 \\
&\bar{V}_o = \bar{g}_{oo} V^o + \bar{g}_{oo} a_\upsilon \bar{V}^\upsilon \Rightarrow V^o = \bar{g}^{oo} \bar{V}_o - a_\upsilon \bar{V}^\upsilon \quad . \\
&\bar{V}^\upsilon = \bar{g}^{\upsilon\upsilon'} V_{\upsilon'} - \bar{g}^{\upsilon\upsilon'} a_{\upsilon'} \bar{V}_o \Rightarrow V_\upsilon = \bar{g}_{\upsilon\upsilon'} \bar{V}^{\upsilon'} + a_\upsilon \bar{V}_o
\end{aligned} \tag{1.1}$$

We start in this frame.

We have found that the following transformation does the job, with $\bar{\phi}\bar{V}^o\bar{V}_o = 1$, where the flows are defined in the *stationary holonomic frame*:

$$
\begin{aligned}
U &= \begin{pmatrix} \bar{V}_o\bar{V}^o\delta^k{}_l & -\bar{V}_o V^k & 0 \\ \bar{V}^o V_l & \bar{V}^o\bar{V}_o & \bar{V}^o\bar{V}_{\upsilon''} \\ 0 & -\bar{V}_o\bar{V}^{\upsilon} & \bar{V}_o\bar{V}^o\delta^{\upsilon}_{\upsilon''} \end{pmatrix} \\
U^{-1} &= \begin{pmatrix} \bar{\phi}\left(\delta^l{}_i - V^l V_i\right) & \bar{\phi}\bar{V}_o V^l & -\bar{\phi}V^l\bar{V}_{\upsilon'} \\ -\bar{\phi}\bar{V}^o V_i & 1 & -\bar{\phi}\bar{V}^o\bar{V}_{\upsilon'} \\ -\bar{\phi}V_i\bar{V}^{\upsilon''} & \bar{\phi}\bar{V}_o\bar{V}^{\upsilon''} & \bar{\phi}\left(\delta^{\upsilon''}_{\upsilon'} - \bar{V}^{\upsilon''}\bar{V}_{\upsilon'}\right) \end{pmatrix}.
\end{aligned}
\tag{1.2}
$$

These matrices are the inverse of each other. We check that they transform the spatial components of the engagement flows to zero:

$$
\begin{aligned}
\tilde{V}^{\nu} &= UV^{\mu} = \begin{pmatrix} \bar{V}_o\bar{V}^o\delta^k{}_l & -\bar{V}_o V^k & 0 \\ \bar{V}^o V_l & \bar{V}^o\bar{V}_o & \bar{V}^o\bar{V}_{\upsilon''} \\ 0 & -\bar{V}_o\bar{V}^{\upsilon} & \bar{V}_o\bar{V}^o\delta^{\upsilon}_{\upsilon''} \end{pmatrix}\begin{pmatrix} V^l \\ \bar{V}^o \\ \bar{V}^{\upsilon''} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{V}^o \\ 0 \end{pmatrix} \\
\tilde{V}_{\mu} &= U^{-1T}V_{\nu} \\
\tilde{V}_{\mu} &= \begin{pmatrix} \bar{\phi}\left(\delta^j{}_k - V^j V_k\right) & -\bar{\phi}\bar{V}^o V_k & -\bar{\phi}V_k\bar{V}^{\upsilon'} \\ \bar{\phi}\bar{V}_o V^j & 1 & \bar{\phi}\bar{V}_o\bar{V}^{\upsilon'} \\ -\bar{\phi}V^j\bar{V}_{\upsilon} & -\bar{\phi}\bar{V}^o\bar{V}_{\upsilon} & \bar{\phi}\left(\delta^{\upsilon'}_{\upsilon} - \bar{V}^{\upsilon'}\bar{V}_{\upsilon}\right) \end{pmatrix}\begin{pmatrix} V_j \\ \bar{V}_o \\ \bar{V}_{\upsilon'} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\phi}\bar{V}_o \\ 0 \end{pmatrix}
\end{aligned}.
$$

The frame so defined is the ***strictly stationary frame***.

1.4 Ownership in the Strictly Stationary Frame

We define ownership as the existence of an ownership structure:

$$\tilde{Q}[J]^{\mu}{}_{\nu} = \begin{pmatrix} J^{j}{}_{k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J^{\upsilon}{}_{\upsilon'} \end{pmatrix}$$
$$J^{j}{}_{k} = \begin{cases} \delta^{j}{}_{k} & j,k \in o[J] \\ 0 & otherwise \end{cases} \quad . \tag{1.3}$$
$$J^{\upsilon}{}_{\upsilon'} = \begin{cases} \delta^{\upsilon}{}_{\upsilon'} & \upsilon,\upsilon' \in o[J] \\ 0 & otherwise \end{cases}$$

Since the only non-zero flow in this frame is along the time direction, the ownership structure is orthogonal to the flow (left and right multiplication). In general, the covariant form of the ownership structure is not a symmetric matrix.

1.5 Ownership in the Stationary Holonomic Frame

We transform back to the orthogonal *stationary frame*. We express the results in terms of the following projection operators for the inactive, active and flow directions:

$$\begin{aligned}
\bar{J}[i]^{\nu}{}_{l} &= \left(\delta^{\nu}_{j'} - \bar{V}^{\nu}V_{j'}\right)J^{j'}{}_{l} = \bar{h}^{\nu}_{j[J]}\delta^{j[J]}{}_{l} \equiv \sum_{j[J]\in o[J]} \bar{h}^{\nu}_{j[J]}\delta^{j[J]}{}_{l} \\
\bar{J}[a]^{\nu}{}_{\upsilon} &= \left(\delta^{\nu}_{\upsilon'} - \bar{V}^{\nu}\bar{V}_{\upsilon'}\right)J^{\upsilon'}{}_{\upsilon} = \bar{h}^{\nu}_{\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon} \equiv \sum_{\upsilon[J]\in o[J]} \bar{h}^{\nu}_{\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon} \\
\bar{J}[o]^{\nu}{}_{o} &= -\bar{\phi}\bar{V}_{o}\left(\left(\delta^{\nu}_{j} - \bar{V}^{\nu}V_{j}\right)J^{j}{}_{k}V^{k} + \left(\delta^{\nu}_{\upsilon} - \bar{V}^{\nu}\bar{V}_{\upsilon}\right)J^{\upsilon}{}_{\upsilon'}\bar{V}^{\upsilon'}\right) \\
\bar{J}[o]^{\nu}{}_{o} &= -\bar{\phi}\bar{V}_{o}\left(\bar{h}^{\nu}_{j[J]}V^{j[J]} + \bar{h}^{\nu}_{\upsilon[J]}\bar{V}^{\upsilon[J]}\right)
\end{aligned} \quad . \tag{1.4}$$

Note the new summation convention for owned strategies. We find that the energy momentum constraints depend on these forms.

We start by transforming the projection operator Eq. (1.3) into the orthogonal *stationary frame*:

$$\bar{Q}[J]^{\mu}{}_{\nu} = U^{-1}\tilde{Q}[J]^{\mu}{}_{\nu}U \tag{1.5}$$

The resultant expression is:

$$\bar{Q}[J]^{\mu}{}_{\nu} = \begin{pmatrix} \bar{J}[i]^{j}{}_{k} & \bar{J}[o]^{j}{}_{o} & \bar{J}[a]^{j}{}_{\upsilon'} \\ \bar{J}[i]^{o}{}_{k} & \bar{J}[o]^{o}{}_{o} & \bar{J}[a]^{o}{}_{\upsilon'} \\ \bar{J}[i]^{\upsilon}{}_{k} & \bar{J}[o]^{\upsilon}{}_{o} & \bar{J}[a]^{\upsilon}{}_{\upsilon'} \end{pmatrix}. \tag{1.6}$$

Note that the terms are evaluated in terms of the tensor components:

$$\left(\gamma_{ji} - V_j V_i\right) J^{i}{}_{k} = \begin{cases} \gamma_{jk} - V_j V_k & k \in o[J] \\ 0 & k \notin o[J] \end{cases}.$$

As such, these expressions depend directly on the scalars and tensors defined in the *stationary holonomic frame*, which are determined by solving the field equations.

1.6 Energy Momentum Tensor

We create the symmetric stress tensor from the projection operator above, assuming a "viscous fluid" of multiple immiscible components:

$$\bar{p}_{\mu\nu} = \sum_J \varphi_J \bar{Q}[J]^{\lambda}{}_{\mu} \bar{g}_{\lambda\rho} \bar{Q}[J]^{\rho}{}_{\nu} - \sum_J \eta_J \bar{Q}[J]^{\lambda}{}_{\mu} \bar{\sigma}_{\lambda\rho} \bar{Q}[J]^{\rho}{}_{\nu}. \tag{1.7}$$

The stress tensor is defined in terms of the energy momentum tensor:

$$T_{\mu\nu} = \mu V_\mu V_\nu - p_{\mu\nu}. \tag{1.8}$$

Of specific interest will be the breakdown of the expression for the stress tensor in components along inactive, active and time directions. To evaluate these expressions, we start with an expression for a general symmetric matric $t_{\mu\nu}$:

$$\bar{t}[J]_{\mu\nu} = \bar{Q}[J]^{\lambda}{}_{\mu} \bar{t}_{\lambda\rho} \bar{Q}[J]^{\rho}{}_{\nu}. \tag{1.9}$$

This evaluates to:

$$\bar{t}\left[J\right]_{\rho\sigma}=\begin{pmatrix} \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{t}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} & \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{t}_{\mu\nu}\bar{J}\left[o\right]^{\nu}{}_{o} & \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{t}_{\mu\nu}\bar{J}\left[a\right]^{\nu}{}_{\upsilon'} \\ \bar{J}\left[o\right]^{\mu}{}_{o}\,\bar{t}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} & \bar{J}\left[o\right]^{\mu}{}_{o}\,\bar{t}_{\mu\nu}\bar{J}\left[o\right]^{\nu}{}_{o} & \bar{J}\left[o\right]^{\mu}{}_{o}\,\bar{t}_{\mu\nu}\bar{J}\left[a\right]^{\nu}{}_{\upsilon'} \\ \bar{J}\left[a\right]^{\mu}{}_{\upsilon}\,\bar{t}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} & \bar{J}\left[a\right]^{\mu}{}_{\upsilon}\,\bar{t}_{\mu\nu}\bar{J}\left[o\right]^{\nu}{}_{o} & \bar{J}\left[a\right]^{\mu}{}_{\upsilon}\,\bar{t}_{\mu\nu}\bar{J}\left[a\right]^{\nu}{}_{\upsilon'} \end{pmatrix}. \tag{1.10}$$

We obtain the player-J contribution to the energy-momentum stress tensor $p_{\mu\nu}$ from Eq. (1.7):

$$\begin{aligned} p\left[J\right]_{\mu\nu} &= Q\left[J\right]^{\rho}{}_{\mu}\,P\left[J\right]_{\rho\sigma}\,Q\left[J\right]^{\sigma}{}_{\nu} \\ P\left[J\right]_{\rho\sigma} &\equiv \varphi_J h_{\rho\sigma}-\eta_J\sigma_{\rho\sigma} \end{aligned}. \tag{1.11}$$

As suggested in Vol. 1, each player is constrained by the stresses for the strategies it owns.

We draw some immediate conclusions from the general form of the stress tensor Eq. (1.8). Based on the properties of the projection operators $Q\left[J\right]$ and it transpose $Q\left[J\right]^{T}$, they are idempotent and orthogonal to the flow, so if we project the energy momentum tensor with these operators we get:

$$\bar{Q}\left[J\right]^{\rho}{}_{\mu}\,\bar{T}_{\rho\sigma}\bar{Q}\left[K\right]^{\sigma}{}_{\nu}=\delta_{JK}\bar{t}\left[J\right]_{\mu\nu}. \tag{1.12}$$

In other words, the energy-momentum tensor is block-diagonal with respect to the player ownership. This supports the idea that player payoffs should have no self-payoffs (Cf. Vol. 2).

We can evaluate the pressure and shear separately, starting with the inactive-inactive contributions:

$$\begin{aligned} \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} &= \delta_{j}{}^{j[J]}\bar{P}\left[J\right]_{j[J]k[J]}\,\delta^{k[J]}{}_{k} \\ \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} &= \varphi_J\delta_{j}{}^{j[J]}h_{j[J]k[J]}\delta^{k[J]}{}_{k}-\eta_J\delta_{j}{}^{j[J]}\sigma_{j[J]k[J]}\delta^{k[J]}{}_{k} \\ \bar{J}\left[i\right]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}\left[i\right]^{\nu}{}_{k} &= \begin{cases} j,k\in o\left[J\right] & \varphi_J\left(\gamma_{jk}-V_jV_k\right)-\eta_J\sigma_{jk} \\ otherwise & 0 \end{cases} \\ J\left[i\right]^{\mu}{}_{j}\,p_{\mu\nu}J\left[i\right]^{\nu}{}_{k} &= \begin{cases} j,k\in o\left[J\right] & \varphi_J\left(\gamma_{jk}-V_jV_k\right)-\eta_J\sigma_{jk} \\ otherwise & 0 \end{cases} \end{aligned}. \tag{1.13}$$

As we expect from Eq. (1.12), only player J stresses occur for the inactive-inactive energy momentum contributions. The orthogonal basis is the same as the holonomic basis.

We expect a similar result for the active-active contributions and inactive-active contributions. For the former we have:

$$\begin{aligned}
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \delta_{\upsilon}{}^{\upsilon[J]}P[J]_{\upsilon[J]\upsilon'[J]}\,\delta^{\upsilon'[J]}{}_{\upsilon'} \\
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \varphi_J\delta_{\upsilon}{}^{\upsilon[J]}\bar{h}_{\upsilon[J]\upsilon'[J]}\delta^{\upsilon'[J]}{}_{\upsilon'} - \eta_J\delta_{\upsilon}{}^{\upsilon[J]}\bar{\sigma}_{\upsilon[J]\upsilon'[J]}\delta^{\upsilon'[J]}{}_{\upsilon'}\,. \\
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \begin{cases} \upsilon,\upsilon'\in o[J] & \varphi_J\left(\bar{g}_{\upsilon\upsilon'} - \bar{V}_{\upsilon}\bar{V}_{\upsilon'}\right) - \eta_J\bar{\sigma}_{\upsilon\upsilon'} \\ otherwise & 0 \end{cases}
\end{aligned} \quad (1.14)$$

The ownership categories in the *stationary holonomic frame* are the same as in the *strictly stationary frame*.

We recover the holonomic form using the contravariant forms:

$$J[a]^{\mu\upsilon}\,\bar{p}_{\mu\nu}J[a]^{\nu\upsilon'} = \varphi_J\,\bar{g}^{\upsilon\upsilon[J]}\bar{h}_{\upsilon[J]\upsilon'[J]}\bar{g}^{\upsilon'[J]\upsilon'} - \eta_J\,\bar{g}^{\upsilon\upsilon[J]}\bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{g}^{\upsilon'[J]\upsilon'}\,. \quad (1.15)$$

We get for the inactive-active contributions:

$$\begin{aligned}
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon} &= \delta_{j}{}^{j[J]}\bar{P}[J]_{j[J]\upsilon[J]}\,\delta^{\upsilon[J]}{}_{\upsilon} \\
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon} &= -\varphi_J\delta_{j}{}^{j[J]}V_{j[J]}\bar{V}_{\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon} - \eta_J\delta_{j}{}^{j[J]}\bar{\sigma}_{j[J]\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon}\,. \\
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon} &= \begin{cases} \mathrm{j},\upsilon\in o[J] & -\varphi_J V_j\bar{V}_{\upsilon} - \eta_J\bar{\sigma}_{j\upsilon} \\ otherwise & 0 \end{cases}
\end{aligned} \quad (1.16)$$

Again, we recover the holonomic form using the contravariant form:

$$J[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}J[a]^{\nu\upsilon} = -\varphi_J\delta_{j}{}^{j[J]}V_{j[J]}\bar{V}_{\upsilon[J]}\bar{g}^{\upsilon[J]\upsilon} - \eta_J\delta_{j}{}^{j[J]}\bar{\sigma}_{j[J]\upsilon[J]}\bar{g}^{\upsilon[J]\upsilon}\,. \quad (1.17)$$

The time contributions are a little different for the inactive-time and active-time contributions. For the former we have:

$$\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = -\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\bar{P}[J]_{j[J]\sigma}\left(\delta^{\sigma}_{k[J]}V^{k[J]} + \delta^{\sigma}_{\upsilon[J]}\bar{V}^{\upsilon[J]}\right)$$
$$\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = -\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\left(\bar{P}[J]_{j[J]k[J]}V^{k[J]} + \bar{P}[J]_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\right)$$
$$\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = -\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\begin{pmatrix}\varphi_{J}h_{j[J]k[J]}V^{k[J]} - \eta_{J}\sigma_{j[J]k[J]}V^{k[J]} \\ +\varphi_{J}\bar{h}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]} - \eta_{J}\bar{\sigma}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\end{pmatrix} \quad . \quad (1.18)$$
$$\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\begin{pmatrix}\varphi_{J}\left(V_{j[J]}\bar{\Psi}_{J} - \gamma_{j[J]k[J]}V^{k[J]}\right) \\ +\eta_{J}\left(\sigma_{j[J]k[J]}V^{k[J]} + \bar{\sigma}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\right)\end{pmatrix}$$

We find it convenient in the last equation to define the scalar:

$$\bar{\Psi}_{J} \equiv V_{k[J]}V^{k[J]} + \bar{V}_{\upsilon[J]}\bar{V}^{\upsilon[J]}. \quad (1.19)$$

For the active-time contributions we have:

$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = -\delta^{\upsilon[J]}{}_{\upsilon}P[J]_{\upsilon[J]\sigma}\bar{\phi}\bar{V}_{o}\left(\delta^{\sigma}_{j[J]}V^{j[J]} + \delta^{\sigma}_{\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)$$
$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_{o}\delta^{\upsilon[J]}{}_{\upsilon}\left(-\bar{P}[J]_{\upsilon[J]j[J]}V^{j[J]} - \bar{P}[J]_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right). \quad (1.20)$$
$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_{o}\delta^{\upsilon[J]}{}_{\upsilon}\begin{pmatrix}\varphi_{J}\left(\bar{V}_{\upsilon[J]}\bar{\Psi}_{J} - \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right) \\ +\eta_{J}\left(\bar{\sigma}_{\upsilon[J]j[J]}V^{j[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\end{pmatrix}$$

For the holonomic form, we use the contravariant expressions:

$$J[a]^{\mu\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_{o}\bar{g}^{\upsilon\upsilon[J]}\begin{pmatrix}\varphi_{J}\left(\bar{V}_{\upsilon[J]}\bar{\Psi}_{J} - \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right) \\ +\eta_{J}\left(\bar{\sigma}_{\upsilon[J]j[J]}V^{j[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\end{pmatrix}. \quad (1.21)$$

Finally, we have the time-time contributions:

$$
\begin{aligned}
&\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \left(\bar{\phi}\bar{V}_{o}\right)^{2}\left(\delta^{\rho}_{j[J]}V^{j[J]} + \delta^{\rho}_{\upsilon[J]}\bar{V}^{\upsilon[J]}\right)\bar{P}[J]_{\rho\sigma}\left(\delta^{\sigma}_{k[J]}V^{k[J]} + \delta^{\sigma}_{\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\\
&\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \left(\bar{\phi}\bar{V}_{o}\right)^{2}\begin{pmatrix} P[J]_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{P}[J]_{j[J]\upsilon'[J]}V^{j[J]}\bar{V}^{\upsilon'[J]} \\ +P[J]_{\upsilon[J]k[J]}V^{k[J]}\bar{V}^{\upsilon[J]} + \bar{P}[J]_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix}\\
&\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \left(\bar{\phi}\bar{V}_{o}\right)^{2}\begin{pmatrix} \varphi_{J}\left(\gamma_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} - \bar{\psi}_{J}\bar{\psi}_{J}\right) \\ -\eta_{J}\begin{pmatrix} \sigma_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{\sigma}_{j[J]\upsilon'[J]}V^{j[J]}\bar{V}^{\upsilon'[J]} \\ +\bar{\sigma}_{\upsilon[J]k[J]}V^{k[J]}\bar{V}^{\upsilon[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix} \end{pmatrix}
\end{aligned}
\quad . \quad (1.22)
$$

We use these explicit expressions to interpret the energy momentum constraint contributions of the player ownership model.

1.7 Acceleration in the Ownership Model

Of particular interest will be the different forces that occur in the ownership model. We know from Vol. 2 that there will in general be three distinct forces acting for the active strategies: payoff forces, cooperative forces, and inertial forces. We certainly expect the inertial forces to reflect player ownership. We may also expect changes to the other two forces. To investigate this we compute the acceleration for the inactive and active flows. As a warning, the calculations below are of necessity technical, and are not essential for understanding the basic idea of ownership. In the end, the important result for numerical work will be the energy momentum tensor that is obtained from these calculations.

1.7.1 *Longitudinal Conservation*

The acceleration is the covariant rate of change of the flow, which we compute from the conservation law for the energy momentum tensor. There are two sets of conservation laws: one is the longitudinal conservation law:

$$\begin{aligned}
&V^{\upsilon}\partial_{\upsilon}\mu + \left(\mu h_{\mu\lambda} + p_{\mu\lambda}\right)\theta^{\mu\lambda} = 0 \\
&V_{\mu;\lambda}p^{\mu\lambda} = \theta_{\mu\lambda}p^{\mu\lambda} \\
&V_{\mu;\lambda}h^{\mu\lambda} = \theta \\
&V^{\mu}p_{\mu\nu;\lambda}g^{\nu\lambda} = -\theta_{\mu\nu}p^{\mu\nu}
\end{aligned} \quad . \tag{1.23}$$

1.7.2 *Transverse Conservation Law*

The other is the set of transverse conservation laws:

$$\mu\dot{V}_{\mu} = h_{\mu}{}^{\sigma}p_{\sigma\nu;\lambda}g^{\lambda\nu} . \tag{1.24}$$

The two components of interest are the inactive and holonomic sets:

$$\begin{aligned}
&\mu\dot{V}_k = \mu V^{\upsilon}\partial_{\upsilon}V_k \\
&\mu\dot{V}_a = \mu V^{b}\partial_{b}V_a - \tfrac{1}{2}\mu\partial_a g_{bc}V^bV^c - \mu V_k F^k_{ab}V^b - \tfrac{1}{2}\mu\partial_a\gamma_{kj}V^kV^j
\end{aligned} . \tag{1.25}$$

1.7.3 *Transverse Equations in terms of Stresses*

This is in the *holonomic stationary frame*; in the *stationary orthogonal frame*, we obtain

$$\mu\bar{g}_{\upsilon\upsilon'}\dot{V}^{\upsilon'} = \begin{pmatrix}
\mu V^{\upsilon'}\partial_{\upsilon'}\bar{V}_{\upsilon} - \tfrac{1}{2}\mu\partial_{\upsilon}\bar{g}_{oo}\bar{V}^{o}\bar{V}^{o} \\
-\tfrac{1}{2}\mu\partial_{\upsilon}\bar{g}_{\upsilon'\upsilon''}\bar{V}^{\upsilon'}\bar{V}^{\upsilon''} - \tfrac{1}{2}\mu\partial_{\upsilon}\gamma_{kj}V^kV^j \\
-\mu\bar{V}_o\left(\partial_{\upsilon}a_{\upsilon'} - \partial_{\upsilon'}a_{\upsilon}\right)\bar{V}^{\upsilon'} - \mu V_k\left(\bar{F}^k_{\upsilon o}\bar{V}^o + \bar{F}^k_{\upsilon\upsilon'}\bar{V}^{\upsilon'}\right)
\end{pmatrix} . \tag{1.26}$$

We obtain the transverse acceleration Eq. (1.24) for each set:

$$\begin{aligned}
&\mu\dot{V}_{j[J]} = h^{\sigma}{}_{j[J]}p_{\sigma\nu;\lambda}g^{\lambda\nu} = p_{j[J]\nu;\lambda}g^{\lambda\nu} - V_{j[J]}V^{\sigma}p_{\sigma\nu;\lambda}g^{\lambda\nu} \\
&\mu\dot{V}_a = h^{\sigma}{}_a p_{\sigma\nu;\lambda}g^{\lambda\nu} = p_{a\nu;\lambda}g^{\lambda\nu} - V_aV^{\sigma}p_{\sigma\nu;\lambda}g^{\lambda\nu} \\
&V^{\sigma}p_{\sigma\nu;\lambda}g^{\lambda\nu} = V^{j}p_{j\nu;\lambda}g^{\lambda\nu} + V^{a}p_{a\nu;\lambda}g^{\lambda\nu} = -V_{\mu;\lambda}p^{\mu\lambda} = -\theta_{\mu\lambda}p^{\mu\lambda}
\end{aligned} . \tag{1.27}$$

We obtain the form for each in terms of the pressure components:

$$
\begin{aligned}
p_{k\nu;\lambda} g^{\nu\lambda} &= \frac{1}{\sqrt{|g\gamma|}} \partial_a \left(\sqrt{|g\gamma|} g^{ab} p_{kb} \right) \\
p_{a\nu;\lambda} g^{\nu\lambda} &= \frac{1}{\sqrt{|g\gamma|}} \partial_c \left(\sqrt{|g\gamma|} g^{bc} p_{ab} \right) + \tfrac{1}{2} \partial_a g^{bc} p_{bc} + \tfrac{1}{2} \partial_a \gamma^{ij} p_{ij} - F^l_{ab} p_l{}^b
\end{aligned}
\tag{1.28}
$$

1.7.4 *Equations in the Covariant Gauge*

In the covariant gauge, these expressions simplify:

$$
\begin{aligned}
p_{k\nu;\lambda} g^{\nu\lambda} &= g^{ab} \partial_a p_{kb} \\
p_{a\nu;\lambda} g^{\nu\lambda} &= g^{bc} \partial_c p_{ab} + \tfrac{1}{2} \partial_a g^{bc} p_{bc} + \tfrac{1}{2} \partial_a \gamma^{ij} p_{ij} - F^l_{ab} p_l{}^b
\end{aligned}.
\tag{1.29}
$$

We obtain the acceleration equations from these expressions:

$$
\begin{aligned}
\mu \dot{V}_k &= p_{k\nu;\lambda} g^{\nu\lambda} - V_k V^\mu p_{\mu\nu;\lambda} g^{\nu\lambda} \\
\mu \dot{V}_a &= p_{a\nu;\lambda} g^{\nu\lambda} - V_a V^\mu p_{\mu\nu;\lambda} g^{\nu\lambda} \\
V^\mu p_{\mu\nu;\lambda} g^{\nu\lambda} &= \begin{pmatrix} g^{ab} V^k \partial_a p_{kb} + g^{ab} V^c \partial_a p_{bc} \\ + \tfrac{1}{2} V^c \partial_c g^{ab} p_{ab} + \tfrac{1}{2} V^c \partial_c \gamma^{ij} p_{ij} - V^c F^l_{cb} p_l{}^b \end{pmatrix}
\end{aligned}
\tag{1.30}
$$

We see the qualitative behavior from the first term of the projection operator.

We turn these into expressions for the orthogonal frame:

$$
\begin{aligned}
p_{k\nu;\lambda} g^{\nu\lambda} &= \bar{g}^{\upsilon\upsilon'} \partial_\upsilon \bar{p}_{k\upsilon'} \\
\bar{g}_{\upsilon\upsilon'} p^{\upsilon'}{}_{\nu;\lambda} g^{\nu\lambda} &= \begin{pmatrix} \bar{g}^{\upsilon''\upsilon'} \partial_{\upsilon'} \bar{p}_{\upsilon\upsilon''} - \bar{g}^{\upsilon''\upsilon'} \left(\partial_\upsilon a_{\upsilon'} - \partial_{\upsilon'} a_\upsilon \right) \bar{p}_{o\upsilon''} \\ + \tfrac{1}{2} \partial_\upsilon e^{2\nu} \bar{p}_{oo} + \tfrac{1}{2} \partial_\upsilon \bar{g}^{\upsilon'\upsilon''} \bar{p}_{\upsilon''\upsilon'} \\ + \tfrac{1}{2} \partial_\upsilon \gamma^{ij} p_{ij} - \bar{g}^{oo} \bar{F}^k_{\upsilon o} p_{ko} - \bar{F}^k{}_{\upsilon\upsilon'} \bar{p}_k{}^{\upsilon'} \end{pmatrix}
\end{aligned}.
\tag{1.31}
$$

1.7.5 *Longitudinal Equations*

The longitudinal part is:

$$V^{\mu} p_{\mu\nu;\lambda} g^{\nu\lambda} = \begin{pmatrix} \bar{V}^{\upsilon} \bar{g}^{\upsilon''\upsilon'} \partial_{\upsilon'} \bar{p}_{\upsilon\upsilon''} - \bar{V}^{\upsilon} \bar{g}^{\upsilon''\upsilon'} \left(\partial_{\upsilon} a_{\upsilon'} - \partial_{\upsilon'} a_{\upsilon} \right) \bar{p}_{o\upsilon''} \\ + \tfrac{1}{2} \bar{V}^{\upsilon} \partial_{\upsilon} e^{2\nu} \bar{p}_{oo} + \tfrac{1}{2} \bar{V}^{\upsilon} \partial_{\upsilon} \bar{g}^{\upsilon'\upsilon''} \bar{p}_{\upsilon''\upsilon'} \\ + \tfrac{1}{2} \bar{V}^{\upsilon} \partial_{\upsilon} \gamma^{ij} p_{ij} - \bar{g}^{oo} \bar{V}^{\upsilon} \bar{F}^{k}_{\upsilon o} \bar{p}_{ko} - \bar{V}^{\upsilon} \bar{F}^{k}_{\upsilon\upsilon'} \bar{p}_{k}{}^{\upsilon'} \\ + \bar{g}^{\upsilon\upsilon'} V^{k} \partial_{\upsilon} \bar{p}_{k\upsilon'} + \bar{g}^{\upsilon''\upsilon'} \bar{V}^{o} \partial_{\upsilon'} \bar{p}_{o\upsilon''} - \bar{V}^{o} \bar{F}^{l}_{o\upsilon} \bar{p}_{l}{}^{\upsilon} \end{pmatrix}. \quad (1.32)$$

1.7.6 *Ownership Equations for Inactive Flow*

By following these technical steps and working in the covariant gauge, we achieve our desired results, starting with an expression for the inactive flow with the ownership attributes identified:

$$\mu \bar{V}^{\upsilon} \partial_{\upsilon} V_{j[J]} = \bar{g}^{\upsilon\upsilon[J]} \partial_{\upsilon} \bar{P}\left[J\right]_{j[J]\upsilon[J]} + V_{j[J]} \theta_{\nu\lambda} p^{\nu\lambda}. \quad (1.33)$$

1.7.7 *Ownership Equations for Active Flow*

The active flow equations are similar:

$$\mu \bar{g}_{\upsilon[J]\upsilon'} \dot{V}^{\upsilon'} = \bar{g}_{\upsilon[J]\upsilon'} p^{\upsilon'}{}_{\nu;\lambda} g^{\nu\lambda} + \bar{V}_{\upsilon[J]} \theta_{\nu\lambda} p^{\nu\lambda}$$

$$\mu \bar{V}^{\upsilon'} \partial_{\upsilon'} \bar{V}_{\upsilon} = \begin{pmatrix} \bar{g}^{\upsilon''\upsilon'} \partial_{\upsilon'} \bar{p}_{\upsilon\upsilon''} - \left(\partial_{\upsilon} a_{\upsilon'} - \partial_{\upsilon'} a_{\upsilon} \right) \bar{p}_{o\upsilon''} \bar{g}^{\upsilon''\upsilon'} \\ + \tfrac{1}{2} \partial_{\upsilon} e^{2\nu} \bar{p}_{oo} + \tfrac{1}{2} \partial_{\upsilon} \bar{g}^{\upsilon'\upsilon''} \bar{p}_{\upsilon''\upsilon'} \\ + \tfrac{1}{2} \partial_{\upsilon} \gamma^{ij} p_{ij} - \bar{g}^{oo} \bar{F}^{k}_{\upsilon o} p_{ko} - \bar{F}^{k}_{\upsilon\upsilon'} \bar{p}_{k}{}^{\upsilon'} \\ + \tfrac{1}{2} \mu \partial_{\upsilon} \bar{g}_{oo} \bar{V}^{o} \bar{V}^{o} + \bar{V}_{\upsilon} \theta_{\nu\lambda} p^{\nu\lambda} \\ + \tfrac{1}{2} \mu \partial_{\upsilon} \bar{g}_{\upsilon'\upsilon''} \bar{V}^{\upsilon'} \bar{V}^{\upsilon''} + \tfrac{1}{2} \mu \partial_{\upsilon} \gamma_{kj} V^{k} V^{j} \\ + \mu \bar{V}_{o} \left(\partial_{\upsilon} a_{\upsilon'} - \partial_{\upsilon'} a_{\upsilon} \right) \bar{V}^{\upsilon'} \\ + \mu V_{k} \left(\bar{F}^{k}_{\upsilon o} \bar{V}^{o} + \bar{F}^{k}_{\upsilon\upsilon'} \bar{V}^{\upsilon'} \right) \end{pmatrix}. \quad (1.34)$$

We expand this expression somewhat and collect terms; the first set of terms mirrors the first set of terms above for the inactive flow:

$$\mu V^{b} \partial_{b} V_{\upsilon[J]} = g^{\upsilon'\upsilon''[J]} \partial_{\upsilon'} \bar{p}_{\upsilon[J]\upsilon''[J]} + \mathcal{C}_{\upsilon[J]} + \mathcal{R}_{\upsilon[J]}. \quad (1.35)$$

We have introduced the following for the new cooperative and payoff terms:

$$\begin{aligned}
&f_{\upsilon\upsilon'} \equiv \partial_{\upsilon} a_{\upsilon'} - \partial_{\upsilon'} a_{\upsilon} \\
&\mathscr{C}_{\upsilon[J]} \equiv \begin{pmatrix} \left(\bar{g}^{oo}\bar{p}_{oo} - \mu\bar{V}_o\bar{V}^o\right)\partial_{\upsilon[J]}\nu \\ +\tfrac{1}{2}\partial_{\upsilon[J]}\bar{g}^{\upsilon'\upsilon''}\left(\bar{p}_{\upsilon'\upsilon''} - \mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''}\right) \\ +\tfrac{1}{2}\partial_{\upsilon[J]}\gamma^{ij}\left(\bar{p}_{ij} - \mu V_i V_j\right) + \bar{V}_{\upsilon[J]}\theta_{\nu\lambda}p^{\nu\lambda} \end{pmatrix}. \\
&\mathscr{R}_{\upsilon[J]} \equiv \begin{pmatrix} f_{\upsilon[J]\upsilon'}\left(\mu\bar{V}_o\bar{V}^{\upsilon'} - \bar{p}_{o\upsilon''}\bar{g}^{\upsilon''\upsilon'}\right) \\ +\bar{F}^k_{\upsilon[J]\upsilon'}\left(\mu V_k\bar{V}^{\upsilon'} - \bar{p}_{k\upsilon''}\bar{g}^{\upsilon''\upsilon'}\right) \\ +\bar{F}^k_{\upsilon[J]o}\left(\mu V_k\bar{V}^o - \bar{g}^{oo}p_{ko}\right) \end{pmatrix}
\end{aligned} \qquad (1.36)$$

1.7.8 *Ownership Properties of the Active Flow*

The inactive contribution to this cooperative force demonstrates the ownership forces:

$$\tfrac{1}{2}\sum_K\left(\left(\varphi_K h_{k[K]k'[K]} - \eta_K\sigma_{k[K]k'[K]}\right)\partial_{\upsilon[J]}\gamma^{k[K]k'[K]} - \mu V_{k[K]}V_j\partial_{\upsilon[J]}\gamma^{k[K]j}\right). \quad (1.37)$$

The forces depend on non-zero gradients of the inactive metric. Thus if the inactive metric is a constant, these contributions are zero.

We have a similar effect for the active metric (gravitational) forces:

$$\begin{pmatrix} \left(\bar{g}^{oo}\bar{p}_{oo} - \mu\bar{V}_o\bar{V}^o\right)\partial_{\upsilon[J]}\nu \\ +\tfrac{1}{2}\partial_{\upsilon[J]}\bar{g}^{\upsilon'\upsilon''}\left(\sum_K\bar{P}_{\upsilon'\upsilon''}[K] - \mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''}\right) \end{pmatrix}. \quad (1.38)$$

If the spatial and mixed components of the active metric are constant, then the only contribution comes from the gradient of the “gravitational field” g^{oo}.

In addition, and of particular interest for our numerical work, there are payoff forces, which are determined from Eq. (1.36):

$$\mathscr{R}_{\upsilon[J]} = \sum_K \begin{pmatrix} \overline{V}_o f_{\upsilon[J]\upsilon'} \left(\mu \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} - \overline{\phi}\varphi_K \left(\mathcal{S}_{\mathbf{K}}{}^{\upsilon'} \overline{\psi}_K - \overline{g}^{\upsilon'\upsilon[K]} \overline{g}_{\upsilon[K]\upsilon'[K]} \overline{V}^{\upsilon'[K]} \right) \right) \\ +\left(\mu + \varphi_K \right) V_{k[K]} \overline{F}^{k[K]}_{\upsilon[J]\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} + \mu V_{k[K]} \overline{F}^{k[K]}_{\upsilon[J]\upsilon'} \sum_{L \neq K} \mathcal{S}_{\mathbf{L}}{}^{\upsilon'} \\ +\overline{F}^{k[K]}_{\upsilon[J]o} \overline{V}^o \left(\mu V_{k[K]} - \overline{\phi}\varphi_K \left(V_{k[K]} \overline{\psi}_K - \gamma_{k[K]k'[K]} V^{k'[K]} \right) \right) \end{pmatrix} \tag{1.39}$$

For those strategies not associated with player-K, we get a current that depends on the energy density only: the current is less and is not viscous. The interesting effects occur within the player's strategies. The ownership model behavior modifies what one would expect from the ideal fluid model.

Similarly for the charge density, the player stresses determine the charges for each player. Only the energy density and the time component of the flow provide effects from other players. Again this differs from the ideal fluid.

The last piece of the puzzle is the contribution to the acceleration itself. We look in the covariant gauge at the active flow contribution for the non-viscous part:

$$\begin{gathered} \frac{1}{\sqrt{|g\gamma|}} \partial_\upsilon \left(\sqrt{|g\gamma|} g^{\upsilon\upsilon'[J]} P[J]_{\upsilon[J]\upsilon'[J]} \right) = g^{\upsilon\upsilon'[J]} \partial_\upsilon \left(\varphi_J h_{\upsilon[J]\upsilon'[J]} \right) \\ g^{\upsilon\upsilon'[J]} \partial_\upsilon \left(\varphi_J h_{\upsilon[J]\upsilon'[J]} \right) = -\varphi_J g^{\upsilon\upsilon'[J]} V_{\upsilon'[J]} \partial_\upsilon V_{\upsilon[J]} + \ldots \end{gathered} \tag{1.40}$$

We see that this enhances the flow on the left-hand side of Eq. (1.35) by a term that depends on the player-J stress, but it is different from an ideal fluid because of the player dependency.

These arguments are still at a high level and need to be brought down to the level of implementing numerical models. In the next section we consider a simplified model that makes the characteristics of player ownership more visible.

1.8 Ideal Ownership Model

In analogy to the ideal fluid, we introduce the ***ideal ownership model*** for the purposes of computing flow ***streamlin***es to get a feeling for the behavior of such models. We intend to use this model for numerical calculations and, with this in mind, we make a number of simplifying assumptions. The purpose of these assumptions is to gain more insight into the model as well as to provide a model that has realistic attributes.

We assume there is no viscosity term; we allow Coriolis forces so that the time component of the metric is $g^{\upsilon o} = -\bar{g}^{\upsilon\upsilon'} a_{\upsilon'}$; we assume that the player stresses are given functions; we assume that the active and inactive metric components are constant in the orthogonal frame. We return at a later date to these equations to identify and remove any errors that may have crept in.

1.8.1 *Stationary Orthogonal Frame Attributes*

We reflect the possibility of stress dependence and those frame dependences that depend on the time component of the metric. The ***stationary orthogonal frame*** has the following attributes:

$$
\begin{aligned}
& g_{ab} g^{bc} = \bar{g}_{ab} \bar{g}^{bc} = \delta_a^{\ c}, \ \bar{g}_{o\upsilon} = 0 \\
& \bar{g}_{oo} = g_{oo}, \ g^{\upsilon\upsilon'} = \bar{g}^{\upsilon\upsilon'} \Rightarrow \bar{g}^{oo} \bar{g}_{oo} = 1 \\
& g_{oo} \equiv e^{-2\nu}, \ g^{o\upsilon} \equiv -g^{\upsilon\upsilon'} a_{\upsilon'} \\
& 0 = g_{o\upsilon'} \bar{g}^{\upsilon'\upsilon} - \bar{g}_{oo} \bar{g}^{\upsilon\upsilon'} a_{\upsilon'} \Rightarrow g_{o\upsilon} = \bar{g}_{oo} a_{\upsilon} \\
& 1 = g_{o\upsilon} g^{\upsilon o} + g_{oo} g^{oo} \Rightarrow g^{oo} = e^{2\nu} + \bar{g}^{\upsilon\upsilon'} a_{\upsilon} a_{\upsilon'} \\
& g_{\upsilon\upsilon''} \bar{g}^{\upsilon''\upsilon'} + g_{\upsilon o} g^{o\upsilon'} = \delta_{\upsilon}^{\ \upsilon'} \Rightarrow g_{\upsilon\upsilon''} = \bar{g}_{\upsilon\upsilon''} + e^{-2\nu} a_{\upsilon} a_{\upsilon''}
\end{aligned}
\quad . \qquad (1.41)
$$

We assume that the spatial metric components $g^{\upsilon\upsilon'}$ are block-diagonal constants in the player space. This implies that the gauge invariant metric components $\bar{g}^{\upsilon\upsilon'}$, $\bar{g}_{\upsilon\upsilon'}$ are constants and are block-diagonal.

We term the metric with a bar, the gauge invariant metric in the *stationary orthogonal frame*. The gauge is based on transformations of the vector a_{υ} (see Vol. 2). In the covariant gauge, these vectors are constrained by $\bar{g}^{\upsilon\upsilon'} \partial_{\upsilon} a_{\upsilon'} = 0$.

We need to compute the transformations between this metric and the original. The following formulas may be useful:

$$
\begin{aligned}
&\overline{V}^{\upsilon} = V^{\upsilon} = \overline{g}^{\upsilon\upsilon'} V_{\upsilon'} - \overline{g}^{\upsilon\upsilon'} a_{\upsilon'} V_o \Rightarrow \overline{V}_{\upsilon} = \overline{g}_{\upsilon\upsilon'} \overline{V}^{\upsilon'} = V_{\upsilon} - a_{\upsilon} V_o \\
&V_{\upsilon} = \overline{V}_{\upsilon} + a_{\upsilon} V_o \\
&\overline{V}_o = V_o = \overline{g}_{oo} V^o + \overline{g}_{oo} a_{\upsilon} \overline{V}^{\upsilon} \Rightarrow \overline{g}^{oo} \overline{V}_o \Rightarrow V^o = \overline{V}^o - a_{\upsilon} \overline{V}^{\upsilon} \\
&\mathcal{S}_{\mathbf{J}}^{\upsilon} \equiv \overline{g}^{\upsilon\upsilon[J]} \overline{V}_{\upsilon[J]} = g^{\upsilon\upsilon[J]} V_{\upsilon[J]} - g^{\upsilon\upsilon[J]} a_{\upsilon[J]} V_o \\
&\sum_J \mathcal{S}_{\mathbf{J}}^{\upsilon} = g^{\upsilon\upsilon'} V_{\upsilon'} - g^{\upsilon\upsilon'} a_{\upsilon'} V_o = g^{\upsilon\upsilon'} V_{\upsilon'} + g^{\upsilon o} V_o = V^{\upsilon} \\
&V_o V^o = 1 - \sum_K \psi_K = \overline{V}_o \overline{V}^o - a_{\upsilon} \overline{V}^{\upsilon} \overline{V}_o \qquad . \qquad (1.42) \\
&\overline{\phi} \overline{V}_o \overline{V}^o \equiv 1 \\
&\overline{\phi} = \overline{\phi} \sum_L \overline{\psi}_L + \overline{\phi} \overline{V}^o \overline{V}_o = \overline{\phi} \sum_L \overline{\psi}_L + 1 \\
&1 = \phi' V^o V_o = \phi' \overline{V}^o \overline{V}_o - \phi' \overline{V}_o a_{\upsilon} \overline{V}^{\upsilon} \Rightarrow \overline{\phi} = \phi' \left(1 - \overline{\phi} \overline{V}_o a_{\upsilon} \overline{V}^{\upsilon}\right) \\
&\phi' = \frac{\overline{\phi}}{1 - \overline{\phi} a_{\upsilon} \overline{V}^{\upsilon} \overline{V}_o} \Rightarrow \frac{1}{\phi'} = \frac{1}{\overline{\phi}} - a_{\upsilon} \overline{V}^{\upsilon} \overline{V}_o
\end{aligned}
$$

1.8.2 *Player Streamlines*

For the ideal fluid, there is a single streamline and it follows the coordinate curve of the energy flow. For the ownership model, we have an active strategy coordinate flow for each player, defined as

$$\mathcal{S}_{\mathbf{J}}^{\upsilon} \equiv \overline{g}^{\upsilon\upsilon[J]} \overline{V}_{\upsilon[J]} . \qquad (1.43)$$

The active strategy flow V^{υ} can be decomposed into these flows:

$$\overline{V}^{\upsilon} = \sum_J \overline{g}^{\upsilon\upsilon'[J]} V_{\upsilon'[J]} = \sum_J \mathcal{S}_{\mathbf{J}}^{\upsilon} . \qquad (1.44)$$

Based on these ***player streamlines***, there will be coordinate curves defined with a parameter s_J for each player:

$$\frac{\partial}{\partial s_J} \equiv \mathcal{S}_{\mathbf{J}}{}^{v}\partial_{v}\,. \tag{1.45}$$

To get an initial sense of the solution behaviors, we assume that the player streamlines commute, which in general would not be true; we ignore curvature effects. A treatment using a full solution to the partial differential equations for the longitudinal and transverse conservation laws will remove this assumption.

Another simplifying assumption is to assume that the divergence of the player streamline vector fields is zero:

$$\partial_{v}\mathcal{S}_{\mathbf{J}}{}^{v} = 0\,. \tag{1.46}$$

This is analogous to assuming that the compression $\theta = 0$ in the ideal fluid, which in fact will also be true the player ownership model as a consequence of the above.

1.8.3 *Energy density, pressure and* α

Finally, since the player stress scalars φ_J are given, the energy density is computed from the conservation law Eq. (1.23). It may be convenient to define the energy density scaled by the average pressure:

$$\alpha \equiv \frac{\mu}{p}\,. \tag{1.47}$$

1.8.4 *Inactive Flow Equation for Ideal Ownership Model*

We are now in a position to get a more transparent form for the transverse and longitudinal equations. We compute the inactive flow equation from Eq. (1.33):

$$\left(\mu+\varphi_J\right)\frac{dV_{j[J]}}{ds_J}+\sum_{K\neq J}\mu\frac{dV_{j[J]}}{ds_K} = -V_{j[J]}\frac{d\varphi_J}{ds_J}+V_{j[J]}\theta_{v\lambda}p^{v\lambda}\,. \tag{1.48}$$

We have a partial differential equation for the coordinate curves of the inactive flow given a choice of stresses for each of the players. We will provide an expression for $\theta_{\nu\lambda} p^{\nu\lambda}$ below. The distinctively new feature is the need to consider multiple coordinate curves. We have a ***mesh-line*** solution as opposed to a streamline solution. For a given player-J, there are different weights depending on the actual values of the player stresses.

1.8.5 *Active Flow Equations for Ideal Ownership Model*

We expect that the active strategies will be more complicated. The behaviors are determined from the active flow Eq. (1.35)

$$\mu \sum_K \frac{\partial V_{\upsilon[J]}}{\partial s_K} + \varphi_J \frac{\partial V_{\upsilon[J]}}{\partial s_J} = \begin{pmatrix} \left(\delta^{\upsilon'}_{\upsilon[J]} - \bar{V}_{\upsilon[J]} \mathcal{S}_{\mathbf{J}}{}^{\upsilon'}\right) \partial_{\upsilon'} \varphi_J \\ \left(\bar{g}^{oo} \bar{p}_{oo} - \mu \bar{V}_o \bar{V}^o\right) \partial_{\upsilon[J]} \nu + \bar{V}_{\upsilon[J]} \theta_{\nu\lambda} p^{\nu\lambda} \\ + f_{\upsilon[J]\upsilon'} \left(\mu \bar{V}_o \bar{V}^{\upsilon'} - \bar{p}_{o\upsilon''} \bar{g}^{\upsilon''\upsilon'}\right) \\ + \bar{F}^k_{\upsilon[J]\upsilon'} \left(\mu V_k \bar{V}^{\upsilon'} - \bar{p}_{k\upsilon''} \bar{g}^{\upsilon''\upsilon'}\right) \\ + \bar{F}^k_{\upsilon[J]o} \left(\mu V_k \bar{V}^o - \bar{g}^{oo} p_{ko}\right) \end{pmatrix}. \quad (1.49)$$

We have defined the effective payoffs as

$$\mu \sum_K \frac{\partial V_{\upsilon[J]}}{\partial s_K} + \varphi_J \frac{\partial V_{\upsilon[J]}}{\partial s_J} = \begin{pmatrix} \left(\delta^{\upsilon'}_{\upsilon[J]} - \bar{V}_{\upsilon[J]} \mathcal{S}_{\mathbf{J}}{}^{\upsilon'}\right) \partial_{\upsilon'} \varphi_J + \bar{V}_{\upsilon[J]} \theta_{\nu\lambda} p^{\nu\lambda} \\ + \left(\bar{g}^{oo} \bar{p}_{oo} - \mu \bar{V}_o \bar{V}^o\right) \partial_{\upsilon[J]} \nu + \bar{V}_o f_{\upsilon[J]\upsilon'} \sum_K \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} \left(\mu + \bar{\phi} \varphi_K \left(1 - \bar{\psi}_K\right)\right) \\ + \sum_K V_{k[K]} \bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'} \left(\mu \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} + \varphi_K \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} + \sum_{L \neq K} \mu \mathcal{S}_{\mathbf{L}}{}^{\upsilon'}\right) \\ + \sum_K \bar{V}^o V_{k[K]} \bar{F}^{k[K]}{}_{\upsilon[J]o} \left(\mu + \bar{\phi} \varphi_K \left(1 - \bar{\psi}_K\right)\right) \end{pmatrix}. \quad (1.50)$$

The ownership forces are clearly displayed. The key elements are analogous but not identical to the ideal fluid. The acceleration along each mesh line is determined by an inertial force based on the gradient of the stress as well as by payoff flows which reflect the character of how the players interact.

1.8.6 *Reduction of Active Flow Equations*

We can do a little more. We can take the next to the last term and write it in terms of the J components and those that are not:

$$\begin{aligned}
&\sum_K V_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'}\left(\mu\mathcal{S}_{\mathbf{K}}{}^{\upsilon'}+\varphi_K\mathcal{S}_{\mathbf{K}}{}^{\upsilon'}+\sum_{L\neq K}\mu\mathcal{S}_{\mathbf{L}}{}^{\upsilon'}\right)=\sum_K\mathcal{F}^{\mathbf{K}}{}_{\upsilon[J]\upsilon'}\mathcal{S}_{\mathbf{K}}{}^{\upsilon'}\\
&=\left((\mu+\varphi_1)V_{k[1]}\bar{F}^{k[1]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[2]}\bar{F}^{k[2]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[3]}\bar{F}^{k[3]}{}_{\upsilon[J]\upsilon'}\right)\mathcal{S}_1^{\upsilon'}\\
&+\left((\mu+\varphi_2)V_{k[2]}\bar{F}^{k[2]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[1]}\bar{F}^{k[1]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[3]}\bar{F}^{k[3]}{}_{\upsilon[J]\upsilon'}\right)\mathcal{S}_2^{\upsilon'}\\
&+\left((\mu+\varphi_3)V_{k[3]}\bar{F}^{k[3]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[1]}\bar{F}^{k[1]}{}_{\upsilon[J]\upsilon'}+\mu V_{k[2]}\bar{F}^{k[2]}{}_{\upsilon[J]\upsilon'}\right)\mathcal{S}_3^{\upsilon'}\;.\\
&+\ldots
\end{aligned}\tag{1.51}$$

$$\mathcal{F}^{\mathbf{K}}{}_{\upsilon[J]\upsilon'}=(\mu+\varphi_K)V_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'}+\sum_{L\neq K}\mu V_{l[L]}\bar{F}^{l[L]}{}_{\upsilon[J]\upsilon'}$$

We do the same thing for the next contribution:

$$\begin{aligned}
&\sum_K\bar{V}^o V_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]o}\left(\mu+\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\right)=\\
&=\sum_K V_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]o}\frac{\bar{V}^o\bar{V}_o\mu+\bar{V}^o\bar{V}_o\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)}{\bar{V}_o}\\
&=\sum_K V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]}\frac{\bar{V}^o\bar{V}_o\mu+\varphi_K\left(1-\bar{\psi}_K\right)}{\bar{V}_o}\qquad.\\
&=-\sum_K V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]}\frac{(\mu+\varphi_K)(1-\bar{\psi}_K)-\mu\sum_{L\neq K}\bar{\psi}_L}{\bar{V}_o}
\end{aligned}\tag{1.52}$$

1.8.7 *Longitudinal Equation for Ideal Ownership Model*

The remaining equation is the longitudinal conservation law. We use Eq. (1.30) to compute $\theta_{\nu\lambda}p^{\nu\lambda}$, which is determined by the energy density flows:

$$-\theta_{\nu\lambda}p^{\nu\lambda}=\sum_K\frac{\partial\mu}{\partial s_K}\,.\tag{1.53}$$

We use the longitudinal conservation law:

$$\sum_K \frac{\partial\mu}{\partial s_K} = V^{\mu} p_{\mu\nu;\lambda} g^{\nu\lambda}$$

$$V^{\mu} p_{\mu\nu;\lambda} g^{\nu\lambda} = \begin{pmatrix} \bar{V}^{\upsilon}\bar{g}^{\upsilon''\upsilon'}\partial_{\upsilon'}\bar{p}_{\upsilon\upsilon''} - \bar{V}^{\upsilon}\bar{g}^{\upsilon''\upsilon'} f_{\upsilon\upsilon'}\bar{p}_{o\upsilon''} \\ +\tfrac{1}{2}\bar{V}^{\upsilon}\partial_{\upsilon} e^{2\nu}\bar{p}_{oo} - \bar{g}^{oo}\bar{V}^{\upsilon}\bar{F}^{k}_{\upsilon o}\bar{p}_{ko} - \bar{V}^{\upsilon}\bar{F}^{k}_{\upsilon\upsilon'}\bar{p}_{k}{}^{\upsilon'} \\ +\bar{g}^{\upsilon\upsilon'}V^{k}\partial_{\upsilon}\bar{p}_{k\upsilon'} + \bar{g}^{\upsilon''\upsilon'}\bar{V}^{o}\partial_{\upsilon'}\bar{p}_{o\upsilon''} - \bar{V}^{o}\bar{F}^{l}_{o\upsilon}\bar{p}_{l}{}^{\upsilon} \end{pmatrix}. \quad (1.54)$$

We separate out the strategy and time components:

$$\sum_K \frac{\partial\mu}{\partial s_K} = \sum_K \begin{pmatrix} \bar{g}^{\upsilon\upsilon'[K]} V^{k[K]}\partial_{\upsilon}\bar{p}_{k[K]\upsilon'[K]} + \bar{g}^{\upsilon\upsilon[K]}\bar{V}^{\upsilon'[K]}\partial_{\upsilon}\bar{p}_{\upsilon[K]\upsilon'[K]} \\ +\bar{V}^{\upsilon}\partial_{\upsilon}\nu e^{2\nu}\bar{p}_{oo}[K] - \bar{V}^{\upsilon}\bar{g}^{\upsilon''\upsilon'} f_{\upsilon\upsilon'}\bar{p}_{o\upsilon''}[K] \\ +\bar{g}^{\upsilon[K]\upsilon}\bar{V}^{o}\partial_{\upsilon}\bar{p}_{o\upsilon[K]} - \bar{V}^{o}\bar{F}^{k[K]}_{o\upsilon} g^{\upsilon\upsilon[K]}\bar{p}_{k[K]\upsilon[K]} \\ -\bar{V}^{\upsilon'}\bar{F}^{k[K]}_{\upsilon'\upsilon} g^{\upsilon\upsilon[K]}\bar{p}_{k[K]\upsilon[K]} - \bar{V}^{\upsilon}\bar{F}^{k[K]}_{\upsilon o}\bar{g}^{oo}\bar{p}_{k[K]o} \end{pmatrix}. \quad (1.55)$$

There are now a number of substitutions required.

1.8.8 *Reduction of Longitudinal Equation*

Since there is a fair amount of algebra, we record the intermediate steps:

$$\sum_K \frac{\partial\mu}{\partial s_K} = \sum_K \begin{pmatrix} -\bar{g}^{\upsilon\upsilon'[K]} V^{k[K]}\partial_{\upsilon}\left(\varphi_K V_{k[K]} V_{\upsilon'[K]}\right) \\ +\bar{g}^{\upsilon\upsilon[K]}\bar{V}^{\upsilon'[K]}\partial_{\upsilon}\left(\varphi_K g_{\upsilon[K]\upsilon'[K]} - \varphi_K V_{\upsilon[K]} V_{\upsilon'[K]}\right) \\ +\bar{V}^{\upsilon}\partial_{\upsilon}\nu e^{2\nu}\left(\bar{\phi}\bar{V}_o\right)^2 \varphi_K\left(\gamma_{j[K]k[K]} V^{j[K]} V^{k[K]} + \bar{g}_{\upsilon[K]\upsilon'[K]}\bar{V}^{\upsilon[K]}\bar{V}^{\upsilon'[K]} - \bar{\psi}_K\bar{\psi}_K\right) \\ -\bar{V}^{\upsilon}\bar{g}^{\upsilon[K]\upsilon'} f_{\upsilon\upsilon'}\left(\bar{\phi}\bar{V}_o\varphi_K\left(V_{\upsilon[K]}\bar{\psi}_K - g_{\upsilon[K]\upsilon'[K]} V^{\upsilon'[K]}\right)\right) \\ +\bar{g}^{\upsilon[K]\upsilon}\bar{V}^{o}\partial_{\upsilon}\left(\bar{\phi}\bar{V}_o\varphi_K\left(V_{\upsilon[K]}\bar{\psi}_K - g_{\upsilon[K]\upsilon'[K]} V^{\upsilon'[K]}\right)\right) \\ +\bar{V}^{o}\varphi_K\bar{F}^{k[K]}_{o\upsilon} g^{\upsilon\upsilon[K]} V_{k[K]} V_{\upsilon[K]} \\ +\bar{V}^{\upsilon'}\varphi_K\bar{F}^{k[K]}_{\upsilon'\upsilon} g^{\upsilon\upsilon[K]} V_{k[K]} V_{\upsilon[K]} \\ +\bar{V}^{\upsilon}\bar{F}^{k[K]}_{o\upsilon}\bar{g}^{oo}\bar{\phi}\bar{V}_o\varphi_K\left(V_{k[K]}\bar{\psi}_K - \gamma_{k[K]k'[K]} V^{k'[K]}\right) \end{pmatrix}. \quad (1.56)$$

This is simplified:

$$\sum_K \frac{\partial \mu}{\partial s_K} = \sum_K \left(\begin{array}{l} -V^{k[K]}\partial_\upsilon \left(\varphi_K V_{k[K]} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \right) + \bar{V}^{\upsilon'[K]} \partial_\upsilon \left(\varphi_K \delta^{\upsilon}_{\upsilon'[K]} - \varphi_K V_{\upsilon'[K]} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \right) \\ +\bar{V}^{\upsilon}\partial_\upsilon \nu \bar{\phi} \varphi_K \left(\gamma_{k[K]k'[K]} V^{k[K]} V^{k'[K]} + \bar{g}_{\upsilon[K]\upsilon'[K]} \bar{V}^{\upsilon[K]} \bar{V}^{\upsilon'[K]} - \bar{\psi}_K \bar{\psi}_K \right) \\ +\bar{\phi}\bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right) \bar{V}^{\upsilon} f_{\upsilon\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} \\ -\bar{V}^{o} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \partial_\upsilon \left(\bar{\phi} \bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right)\right) \\ +\varphi_K V_{k[K]} \bar{F}^{k[K]}_{\upsilon'\upsilon} \bar{V}^{\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} + \bar{V}^{o} \varphi_K V_{k[K]} \bar{F}^{k[K]}_{o\upsilon} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \\ +\bar{\phi}\bar{V}^{o} \varphi_K \left(V_{k[K]} \bar{\psi}_K - \gamma_{k[K]k'[K]} V^{k'[K]} \right) \bar{F}^{k[K]}_{o\upsilon} \bar{V}^{\upsilon} \end{array} \right). \quad (1.57)$$

We assume that the divergences of the player flows vanish, so we get:

$$\sum_K \frac{\partial \mu}{\partial s_K} = \sum_K \left(\begin{array}{l} +\varphi_K V_{k[K]} \dfrac{\partial V^{k[K]}}{\partial s_K} + \varphi_K V_{\upsilon'[K]} \dfrac{\partial \bar{V}^{\upsilon'[K]}}{\partial s_K} + \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right) \bar{V}_o \dfrac{\partial \bar{V}^{o}}{\partial s_K} \\ +\bar{\phi}\varphi_K \bar{\psi}_K \left(1-\bar{\psi}_K\right) \bar{V}^{\upsilon} \partial_\upsilon \nu + \bar{\phi}\bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right) \bar{V}^{\upsilon} f_{\upsilon\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} \\ +\varphi_K V_{k[K]} \bar{F}^{k[K]}_{\upsilon'\upsilon} \bar{V}^{\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} + \bar{V}^{o} \varphi_K V_{k[K]} \bar{F}^{k[K]}_{o\upsilon} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \\ -\bar{\phi}\bar{V}^{o} \varphi_K \left(1-\bar{\psi}_K\right) V_{k[K]} \bar{F}^{k[K]}_{o\upsilon} \bar{V}^{\upsilon} \end{array} \right). \quad (1.58)$$

We can obtain the derivative of the time flow from the other flows using:

$$\begin{aligned} &\tfrac{1}{2} \frac{\partial}{\partial s_K} \left(\bar{g}_{oo} \bar{V}^{o} \bar{V}^{o} \right) = \tfrac{1}{2} \frac{\partial \bar{g}_{oo}}{\partial s_K} \bar{V}^{o} \bar{V}^{o} + \frac{\partial \bar{V}^{o}}{\partial s_K} \bar{V}_o \\ &\frac{\partial \bar{V}^{o}}{\partial s_K} \bar{V}_o = \tfrac{1}{2} \frac{\partial}{\partial s_K} \left(\bar{g}_{oo} \bar{V}^{o} \bar{V}^{o} \right) - \tfrac{1}{2} \frac{\partial e^{-2\nu}}{\partial s_K} \bar{V}^{o} \bar{V}^{o} \\ &\frac{\partial \bar{V}^{o}}{\partial s_K} \bar{V}_o = \frac{\partial \nu}{\partial s_K} \bar{V}^{o} \bar{V}_o + \tfrac{1}{2} \frac{\partial}{\partial s_K} \left(1 - \bar{g}_{\upsilon\upsilon'} \bar{V}^{\upsilon} \bar{V}^{\upsilon'} - \gamma_{jk} \bar{V}^{j} \bar{V}^{k} \right) \\ &\bar{V}_o \frac{\partial \bar{V}^{o}}{\partial s_K} = \bar{V}^{o} \bar{V}_o \frac{\partial \nu}{\partial s_K} - \bar{g}_{\upsilon\upsilon'} \bar{V}^{\upsilon} \frac{\partial \bar{V}^{\upsilon'}}{\partial s_K} - \gamma_{jk} \bar{V}^{j} \frac{\partial \bar{V}^{k}}{\partial s_K} \end{aligned} . \quad (1.59)$$

Using this we obtain:

$$\sum_K \frac{\partial\mu}{\partial s_K} = \sum_K \begin{pmatrix} +\varphi_K V_{k[K]} \dfrac{\partial V^{k[K]}}{\partial s_K} - \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right)\bar{V}_j \dfrac{\partial \bar{V}^j}{\partial s_K} \\ +\varphi_K V_{\upsilon'[K]} \dfrac{\partial \bar{V}^{\upsilon'[K]}}{\partial s_K} - \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right)\bar{V}_\upsilon \dfrac{\partial \bar{V}^\upsilon}{\partial s_K} \\ +\varphi_K \left(1-\bar{\psi}_K\right)\dfrac{\partial \nu}{\partial s_K} + \bar{\phi}\varphi_K \bar{\psi}_K \left(1-\bar{\psi}_K\right)\bar{V}^\upsilon \partial_\upsilon \nu \\ +\bar{\phi}\bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right)\bar{V}^\upsilon f_{\upsilon\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon'} \\ +\varphi_K V_{k[K]} \bar{F}^{k[K]}_{\upsilon'\upsilon} \bar{V}^{\upsilon'} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} + \bar{V}^o \varphi_K V_{k[K]} \bar{F}^{k[K]}_{o\upsilon} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \\ -\bar{\phi}\bar{V}^o \varphi_K \left(1-\bar{\psi}_K\right) V_{k[K]} \bar{F}^{k[K]}_{o\upsilon} \bar{V}^\upsilon \end{pmatrix}. \tag{1.60}$$

We have added the gravitational and Coriolis effects through the time metric components. We note that in solving the partial differential equations, for most partial differential solvers, we will have to linearize the equations. We do this on the right-hand-side by keeping the derivative terms while setting the other terms to given functions; we then iterate using the solutions found to modify the given functions.

1.8.9 *Second Reduction of Longitudinal Equation*

We do one more simplification above, assuming the inactive metric is also block-diagonal:

$$\sum_K \frac{\partial\mu}{\partial s_K} = \sum_K \begin{pmatrix} +\varphi_K V_{k[K]} \dfrac{\partial V^{k[K]}}{\partial s_K} - \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right)\bar{V}_j \dfrac{\partial \bar{V}^j}{\partial s_K} \\ +\varphi_K V_{\upsilon'[K]} \dfrac{\partial \bar{V}^{\upsilon'[K]}}{\partial s_K} - \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right)\bar{V}_\upsilon \dfrac{\partial \bar{V}^\upsilon}{\partial s_K} \\ +\varphi_K \left(1-\bar{\psi}_K\right)\dfrac{\partial \nu}{\partial s_K} + \bar{V}^\upsilon \partial_\upsilon \nu \bar{\phi}\varphi_K \bar{\psi}_K \left(1-\bar{\psi}_K\right) \\ -\bar{\phi}\bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right)\mathcal{S}_{\mathbf{K}}{}^{\upsilon} f_{\upsilon\upsilon'} \sum\limits_{L\neq K} \mathcal{S}_{\mathbf{L}}{}^{\upsilon'} - \varphi_K V_{k[K]} \mathcal{S}_{\mathbf{K}}{}^{\upsilon} \bar{F}^{k[K]}{}_{\upsilon\upsilon'} \sum\limits_{L\neq K} \mathcal{S}_{\mathbf{L}}{}^{\upsilon'} \\ -\bar{\phi}\bar{V}^o \varphi_K V_{k[K]} \bar{F}^{k[K]}{}_{o\upsilon} \left(\sum\limits_{L\neq K} \bar{\psi}_L \mathcal{S}_{\mathbf{K}}{}^{\upsilon} + \left(1-\bar{\psi}_K\right) \sum\limits_{L\neq K} \mathcal{S}_{\mathbf{L}}{}^{\upsilon} \right) \end{pmatrix}. \tag{1.61}$$

We presume we know the player stresses and that Eq. (1.61) determines the energy density along the streamline. Finally, we compute the remaining scalar, along with its mesh-line derivatives:

$$\bar{V}^o\bar{V}_o + \sum_K \bar{\psi}_K = 1$$
$$\bar{\phi}\bar{V}^o\bar{V}_o = 1 \Rightarrow 1 = \bar{\phi}\left(1 - \sum_K \bar{\psi}_K\right). \tag{1.62}$$
$$\bar{\phi} = \frac{1}{1 - \sum_K \bar{\psi}_K}$$

We know the metric, so we compute the time components of flow:

$$\bar{V}^o = \sqrt{\bar{g}^{oo}\left(1 - \sum_K \bar{\psi}_K\right)}$$
$$\bar{\phi}\bar{V}^o = \sqrt{\frac{\bar{g}^{oo}}{1 - \sum_K \bar{\psi}_K}} \quad . \tag{1.63}$$
$$\bar{V}_o = \sqrt{\bar{g}_{oo}\left(1 - \sum_K \bar{\psi}_K\right)}$$

We recall that we have set the spatial metric components to be constants, so the mesh-line derivatives become:

$$\bar{g}_{oo}\bar{V}^o\bar{V}^o + \sum_K \bar{\psi}_K = 1$$
$$\bar{V}_o \frac{\partial \bar{V}^o}{\partial s_J} - \bar{V}^o\bar{V}_o \frac{\partial \nu}{\partial s_J} = -½ \sum_K \frac{\partial \bar{\psi}_K}{\partial s_J} . \tag{1.64}$$
$$\bar{\phi}\bar{V}_o \frac{\partial \bar{V}^o}{\partial s_J} = \frac{\partial \nu}{\partial s_J} - ½ \frac{\sum_K \frac{\partial \bar{\psi}_K}{\partial s_J}}{1 - \sum_K \bar{\psi}_K}$$

The equations are well-defined partial differential equations and suitable for numerical models.

1.9 Central Co-Moving Frame

In the special case that the Killing vector is proportional to the flow, $K^{\mu} = \phi V^{\mu}$, the equations simplify. In particular, as shown in Vol. 1, it is known that the compression components vanish. Thus viscosity effects will be absent:

$$\begin{aligned} &\theta = 0 \\ &\sigma_{jk} = 0 \\ &\sigma_{aj} = \theta_{aj} = 0 \\ &\sigma_{ab} = 0 \end{aligned} \tag{1.65}$$

This agrees with the analysis from Vol. 1, taken from (Hawking & Ellis, 1973). Only the isotropic pressure components contribute.

Nevertheless, this is an interesting class of models. For numerical purposes, it is probably the first class to consider since only the pressure enters.

1.10 Bias Flows in the Centrally Co-Moving Frame

In setting up the problem in the centrally co-moving frame, we assumed that the covariant ***seasonal flow*** V_{υ} components are known. The components are given in terms of the metric components $g_{o\upsilon}$:

$$V_{\upsilon} = g_{\upsilon o} V^{o} . \tag{1.66}$$

The ***player engagement*** components are determined by the ***player biases***:

$$\begin{aligned} &V_{j} = \gamma_{jk} A^{k}{}_{o} V^{o} \Rightarrow V^{j} = A^{j}{}_{o} V^{o} \\ &F^{j}{}_{\upsilon o} = \partial_{\upsilon} A^{j}{}_{o} \end{aligned} . \tag{1.67}$$

The time component of flow is determined from these player engagements and the time component of the metric g_{oo}:

$$1 = g_{oo}V^oV^o + \gamma_{jk}V^jV^k . \tag{1.68}$$

1.11 Stationary Ownership Model

The basic idea of the ***stationary ownership model*** is to proceed as outlined in Vol. 2, Chap. 20, generalizing from the central co-moving frame:

(1) Pick a problem for analysis and start with a game theoretic view.

(2) Assume there is a ***stationary holonomic frame*** in which time is inactive and mutually commutes with the player inactive strategies, including any code of conduct strategies. Though not required by the definition of ownership, the stationary aspect may provide useful insights about the theory. Identify the players, including the codes of conduct players. View the problem in the holonomic frame based on these inactive dimensions.

(3) Choose the independent metric components to be the gauge invariant quantities $\{\gamma_{jk} \quad g^{ab} \quad F^j{}_{ab}\}$, (which in a centrally co-moving frame determine *player engagements* V_j and *seasonal flows* V_υ).

(4) Transform to the frame in which the flows V^υ, V^j and charges V_υ, V_j are zero.

(5) Set the values of the energy momentum tensor using the ***player ownership structures*** in this frame (Sec. 6.10, Ex. 40), Eq. (1.3).

(6) Transform the ownership structures back to the *holonomic frame* noting that ownership applies only to the active and inactive strategies, Eq. (1.6), not to time.

(7) Impose the constraint of ownership through the energy momentum tensor, not unlike constraints for immiscible fluids (*Cf.* Sec. 3.11, Ex. 17 and Eq. (3.37)). Though $Q[J]_{\mu\nu}$ is a projection operator and orthogonal to the flow, it is not symmetric, so use the symmetrized form that implements the idea of ***player*** from Vol. 1, Sec. 7.3, Eq. (1.7) to obtain Eq. (1.10).

(8) Set the gauge to the seasonal gauge from Vol. 2, Sec. 6, Eq. (6.98), noting that we need not set $g_{z\tau} = g_{z\upsilon} = 0$, but rather have their values set by the field equations:

$$\partial_z g_{z\tau} = \partial_z g_{zv} = \partial_z g_{zz} = 0. \tag{6.69}$$

(9) Solve the resultant elliptic partial differential field equations in the *stationary holonomic frame*. The equations are now all elliptic since they are independent of time. Use a general purpose solver to focus on solutions rather than techniques. We believe a future version of Mathematica will do this.

(10) Match the boundary condition of the problem under investigation using the harmonic wave equation.

(11) Compare the results of the simulation against observed behaviors to establish the relevance of the model.

1.12 Summary

I have outlined a strategy for computing in decision behaviors in the decision process theory using a specific model which extends ordinary Game Theory. The philosophical foundation is different, so the analysis of a decision process is slightly modified. One of the biggest differences is the distinction between active strategies and code of conduct strategies. This modifies what we mean by payoffs and utilities.

The next step is to apply the ideas from this white paper to specific examples. Obvious choices for these examples will be to repeat the analysis for games already considered in Vol. 2 and identify the differences that arise by explicitly including the concept of ownership.