Stationary Ownership Update

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1.1 Introduction

This white paper is based on an earlier paper (Stationary Ownership Model, v71) that introduced the notion of stationary ownership. In this white paper, we go through and verify the approach, based on a review of the arguments from the previous paper.

We recall that the goal of the ***stationary ownership model*** is to get a form for the energy momentum tensor articulated in the ***stationary holonomic frame***. This is a holonomic frame in which the active strategies commute and are ***holonomic***": a holonomic coordinate is one specified by a scalar field, not unlike a potential, where constant surfaces define each coordinate value. Since the behaviors are stationary, time is inactive and is assumed to mutually commute with the inactive strategies.

We want to set up the problem for the model here in such a way that we can use the resultant equations for the flow vectors in a more general treatment where we solve not only for the flow vectors but for the active and inactive metric components. The natural breakdown we think is to start in the frame in which the inactive worldview strategies are orthogonal to time and the active strategies treated as holonomic variables. This gives a breakdown of the equations of motion with a natural interpretation. We further break time off and treat it as orthogonal to the strategic directions defining a specific meaning to the stationary frame. Ultimately we hope to solve the equations with no further assumptions. This choice, along with the choice of the covariant gauge, specifies the equations.

We will write out the equations in the above gauge, anticipating that the metric equations will be elliptic equations. We then have the challenge of writing out the iterative approach to solve these equations. All of these issues become apparent when we examine the conservation laws for the flows.

In focusing on stationary behaviors, we take our cue from engineering in which they provide a useful guide to complex behaviors. We have in mind AC circuits, which describe behaviors after transients have died out, as a natural way to proceed from DC circuits.

1.2 The Stationary Holonomic Frame

The ***stationary holonomic frame*** is obtained for models in which the inactive player strategies mutually commute. There will always be such a frame. We can choose this frame to be a holonomic basis for the active strategies (and time) with metric g_{ab} and a non-holonomic basis for the inactive dimensions that lead to the metric components γ_{jk} along with the associated payoffs $F^j{}_{ab}$. This decomposition is dealt with in detail in Vol. 2. This decomposition agrees with the way in which we describe the decision process. Our goal will be to have a decision engineering notebook where the user can input different game models and boundary conditions and the Mathematica notebook will compute the solution to the partial differential equations.

The ownership model structure

- Is based on a projection operator
- Is orthogonal to the flow
- Depends on only the above components along with the player engagements V_j and the flow components V_υ.
- Has a projection operator that has an intuitive form in the ***strictly stationary frame*** in which the player flows vanish. In this frame we believe that ownership will have the desired meaning: it is isotropic (the player has no strategic biases on

the owned strategies) and projects out just the active and inactive strategies that are owned.

- The projection operator should be a diagonal matrix with unit elements for each strategy that is in the ownership group for that player class $o[J]$.

1.3 The Transformation U to the Fully Co-Moving Frame

The first step is to linearly transform to a frame in which the contravariant and covariant flows are zero, the ***fully co-moving frame***. We do this in two steps. First we note that there is always a frame, albeit a non-holonomic-frame, in which the time components are orthogonal to the strategies:

$$\begin{aligned}
&\left\{\bar{g}_{o\upsilon}=0,\ \bar{g}_{oo}=g_{oo},\ \bar{g}^{\upsilon\upsilon'}=g^{\upsilon\upsilon'}\right\}\\
&g_{o\upsilon}=\bar{g}_{oo}a_\upsilon\\
&V_o=\bar{V}_o=\bar{g}_{oo}V^o+\bar{g}_{oo}a_\upsilon V^\upsilon \Rightarrow \bar{V}^o=\bar{g}^{oo}\bar{V}_o=V^o+a_\upsilon V^\upsilon\\
&\bar{V}^\upsilon=V^\upsilon=\bar{g}^{\upsilon\upsilon'}V_{\upsilon'}-\bar{g}^{\upsilon\upsilon'}a_{\upsilon'}V_o \Rightarrow \bar{V}_\upsilon=\bar{g}_{\upsilon\upsilon'}V^{\upsilon'}=V_\upsilon-a_\upsilon V_o\,.\\
&\bar{F}^k_{\upsilon\upsilon'}=F^k{}_{\upsilon\upsilon'}+a_\upsilon F^k{}_{\upsilon' o}-a_{\upsilon'}F^k{}_{\upsilon o}\equiv F^k{}_{\upsilon\upsilon'}+G^k{}_{\upsilon\upsilon'}\\
&\bar{F}^k_{o\upsilon}=F^k{}_{o\upsilon}\\
&f_{\upsilon\upsilon'}=\partial_\upsilon a_{\upsilon'}-\partial_{\upsilon'}a_\upsilon
\end{aligned} \tag{1.1}$$

The payoff will pick up a gauge contribution $G^k{}_{\upsilon\upsilon'}$.

We start in this ***time-orthogonal*** frame. The non-holonomic frame has the following two properties: the time component of the flow (which is now inactive) has the same covariant form in both frames; the strategic flow directions have the same contravariant form in both frames.

The following transformation takes us from this *time-orthogonal* frame to the *fully co-moving frame*, with $\bar{\phi}\bar{V}^o\bar{V}_o=1$, where the *time-orthogonal flows* are defined in Eq. (1.1):

$$U = \begin{pmatrix} \bar{V}_o \bar{V}^o \delta^k{}_l & -\bar{V}_o V^k & 0 \\ \bar{V}^o V_l & \bar{V}^o \bar{V}_o & \bar{V}^o \bar{V}_{\upsilon''} \\ 0 & -\bar{V}_o \bar{V}^\upsilon & \bar{V}_o \bar{V}^o \delta^\upsilon_{\upsilon''} \end{pmatrix}. \tag{1.2}$$

The inverse to this transformation is

$$U^{-1} = \begin{pmatrix} \bar{\phi}\left(\delta^l{}_i - V^l V_i\right) & \bar{\phi}\bar{V}_o V^l & -\bar{\phi} V^l \bar{V}_{\upsilon'} \\ -\bar{\phi}\bar{V}^o V_i & 1 & -\bar{\phi}\bar{V}^o \bar{V}_{\upsilon'} \\ -\bar{\phi} V_i \bar{V}^{\upsilon''} & \bar{\phi}\bar{V}_o \bar{V}^{\upsilon''} & \bar{\phi}\left(\delta^{\upsilon''}_{\upsilon'} - \bar{V}^{\upsilon''}\bar{V}_{\upsilon'}\right) \end{pmatrix}. \tag{1.3}$$

$$UU^{-1} = \begin{pmatrix} \delta^k{}_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^\upsilon_{\upsilon'} \end{pmatrix}$$

We check that they transform the spatial components of the engagement flows to zero:

$$\tilde{V}^\nu = UV^\mu = \begin{pmatrix} \bar{V}_o \bar{V}^o \delta^k{}_l & -\bar{V}_o V^k & 0 \\ \bar{V}^o V_l & \bar{V}^o \bar{V}_o & \bar{V}^o \bar{V}_{\upsilon''} \\ 0 & -\bar{V}_o \bar{V}^\upsilon & \bar{V}_o \bar{V}^o \delta^\upsilon_{\upsilon''} \end{pmatrix} \begin{pmatrix} V^l \\ \bar{V}^o \\ \bar{V}^{\upsilon''} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{V}^o \\ 0 \end{pmatrix}$$

$$\tilde{V}_\mu = U^{-1T} V_\nu .$$

$$\tilde{V}_\mu = \begin{pmatrix} \bar{\phi}\left(\delta_k{}^j - V_k V^j\right) & -\bar{\phi} V_k \bar{V}^o & -\bar{\phi} V_k \bar{V}^{\upsilon'} \\ \bar{\phi}\bar{V}_o V^j & 1 & \bar{\phi}\bar{V}_o \bar{V}^{\upsilon'} \\ -\bar{\phi}\bar{V}_\upsilon V^j & -\bar{\phi}\bar{V}_\upsilon \bar{V}^o & \bar{\phi}\left(\delta_\upsilon{}^{\upsilon'} - \bar{V}_\upsilon \bar{V}^{\upsilon'}\right) \end{pmatrix} \begin{pmatrix} V_j \\ \bar{V}_o \\ \bar{V}_{\upsilon'} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\phi}\bar{V}_o \\ 0 \end{pmatrix}$$

The frame is the ***fully co-moving frame***.

1.4 Ownership in the Fully Co-Moving Frame

We defined ownership (vol. 2) as the existence of the ownership structure:

$$\tilde{Q}[J]^{\mu}{}_{\nu} = \begin{pmatrix} J^{j}{}_{k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J^{\upsilon}{}_{\upsilon'} \end{pmatrix}$$
$$J^{j}{}_{k} = \begin{cases} \delta^{j}{}_{k} & j,k \in o[J] \\ 0 & otherwise \end{cases} . \quad (1.4)$$
$$J^{\upsilon}{}_{\upsilon'} = \begin{cases} \delta^{\upsilon}{}_{\upsilon'} & \upsilon,\upsilon' \in o[J] \\ 0 & otherwise \end{cases}$$

Since the only non-zero flow in this frame is along the time direction, the ownership structure is orthogonal to the flow (left and right multiplication). In general, the covariant form of the ownership structure will not be a symmetric matrix.

1.5 Ownership in the Time-Orthogonal Frame

We start by transforming the projection operator Eq. (1.4) into the *time orthogonal frame*:

$$\bar{Q}[J]^{\mu}{}_{\nu} = U^{-1}\tilde{Q}[J]^{\mu}{}_{\nu}U \equiv \begin{pmatrix} \bar{J}[i]^{j}{}_{k} & \bar{J}[o]^{j}{}_{o} & \bar{J}[a]^{j}{}_{\upsilon'} \\ \bar{J}[i]^{o}{}_{k} & \bar{J}[o]^{o}{}_{o} & \bar{J}[a]^{o}{}_{\upsilon'} \\ \bar{J}[i]^{\upsilon}{}_{k} & \bar{J}[o]^{\upsilon}{}_{o} & \bar{J}[a]^{\upsilon}{}_{\upsilon'} \end{pmatrix} \quad (1.5)$$

This generates the following expressions:

$$\bar{Q}[J]^{\mu}{}_{\nu} = \begin{pmatrix} \left(\delta^{j}{}_{i} - V^{j}V_{i}\right)J^{i}{}_{k} & -\bar{\phi}\bar{V}_{o}\left(\delta^{j}{}_{i}J^{i}{}_{k'}V^{k'} - V^{j}V_{i}J^{i}{}_{k'}V^{k'} - V^{j}\bar{V}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'''}\bar{V}^{\upsilon'''}\right) & -V^{j}\bar{V}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'} \\ -\bar{V}^{o}V_{i}J^{i}{}_{k} & V_{i}J^{i}{}_{k'}V^{k'} + \bar{V}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'''}\bar{V}^{\upsilon'''} & -\bar{V}^{o}\bar{V}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'} \\ -\bar{V}^{\upsilon}V_{i}J^{i}{}_{k} & -\bar{V}_{o}\bar{\phi}\left(\delta^{\upsilon}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'''}\bar{V}^{\upsilon'''} - \bar{V}^{\upsilon}\bar{V}_{\upsilon''}J^{\upsilon''}{}_{\upsilon'''}\bar{V}^{\upsilon'''} - \bar{V}^{\upsilon}V_{i}J^{i}{}_{k'}V^{k'}\right) & \left(\delta^{\upsilon}_{\upsilon''} - \bar{V}^{\upsilon}\bar{V}_{\upsilon''}\right)J^{\upsilon''}{}_{\upsilon'} \end{pmatrix} . \quad (1.6)$$

The relevant components for the inactive, active and time components in the *time orthogonal frame* are:

$$
\begin{aligned}
&\bar{J}[i]^{\mu}{}_{k} = \left(\delta^{\mu}_{j} - \bar{V}^{\mu} V_{j}\right) J^{j}{}_{k} = \bar{h}^{\mu}{}_{j[J]} \delta^{j[J]}{}_{k} \equiv \sum_{j[J]\in o[J]} \bar{h}^{\mu}{}_{j[J]} \delta^{j[J]}{}_{k} \\
&\bar{J}[a]^{\mu}{}_{\upsilon} = \left(\delta^{\mu}_{\upsilon'} - \bar{V}^{\mu} \bar{V}_{\upsilon'}\right) J^{\upsilon'}{}_{\upsilon} = \bar{h}^{\mu}{}_{\upsilon[J]} \delta^{\upsilon[J]}{}_{\upsilon} \equiv \sum_{\upsilon[J]\in o[J]} \bar{h}^{\mu}{}_{\upsilon[J]} \delta^{\upsilon[J]}{}_{\upsilon} \\
&\bar{J}[o]^{\mu}{}_{o} = -\bar{\phi}\bar{V}_{o} \left(\left(\delta^{\mu}_{j} - \bar{V}^{\mu} V_{j}\right) J^{j}{}_{k} V^{k} + \left(\delta^{\mu}_{\upsilon} - \bar{V}^{\mu} \bar{V}_{\upsilon}\right) J^{\upsilon}{}_{\upsilon'} \bar{V}^{\upsilon'} \right) \\
&\bar{J}[o]^{\mu}{}_{o} = -\bar{\phi}\bar{V}_{o} \left(\bar{h}^{\mu}{}_{j[J]} V^{j[J]} + \bar{h}^{\mu}{}_{\upsilon[J]} \bar{V}^{\upsilon[J]} \right) \equiv -\bar{\phi}\bar{V}_{o} \bar{h}^{\mu}{}_{\lambda[J]} V^{\lambda[J]}
\end{aligned}
\quad . \tag{1.7}
$$

Note the new summation convention for owned strategies. We find that the energy momentum constraints depend on these forms. These expressions depend directly on the scalars and tensors defined in the *time orthogonal frame*, which are determined by solving the field equations.

Note that the basic form for the projection operator for directions transverse to the flow is:

$$
\nu \neq o \Rightarrow \bar{J}^{\mu}{}_{\nu} = \bar{h}^{\mu}{}_{\lambda[J]} \delta^{\lambda[J]}{}_{\nu} . \tag{1.8}
$$

The expression holds for any value of $\mu \in \{j,\ o,\ \upsilon\}$ so we can lower the first index. Because we are in the time orthogonal frame, we can raise the second index using only the transverse components. Thus we have:

$$
\begin{aligned}
&\bar{J}_{\mu\nu} = \bar{h}_{\mu\lambda[J]} \delta^{\lambda[J]}{}_{\nu} \\
&\nu \neq o \Rightarrow \bar{J}^{\mu\nu} = \bar{h}^{\mu}{}_{\lambda[J]} \bar{g}^{\lambda[J]\nu}
\end{aligned}
\quad . \tag{1.9}
$$

It is also worth noting that if the covariant metric $\bar{g}_{\upsilon\upsilon'}$ is block-diagonal in the transverse strategy space, then so is the contravariant metric.

1.6 Energy Momentum Tensor

We create the symmetric stress tensor from the projection operator above, assuming a "viscous fluid" of multiple immiscible components in the *time orthogonal frame*:

$$
\bar{p}_{\mu\nu} = \sum_{J} \varphi_{J} \bar{Q}[J]^{\lambda}{}_{\mu} \bar{g}_{\lambda\rho} \bar{Q}[J]^{\rho}{}_{\nu} - \sum_{J} \eta_{J} \bar{Q}[J]^{\lambda}{}_{\mu} \bar{\sigma}_{\lambda\rho} \bar{Q}[J]^{\rho}{}_{\nu} . \tag{1.10}
$$

The stress tensor is defined in terms of the energy momentum tensor:

$$\bar{T}_{\mu\nu} = \mu \bar{V}_\mu \bar{V}_\nu - \bar{p}_{\mu\nu} . \tag{1.11}$$

Of specific interest will be the breakdown of the expression for the stress tensor in components along inactive, active and time directions. To evaluate these expressions, we start with an expression for a general symmetric matric $t_{\mu\nu}$:

$$\bar{t}\left[J\right]_{\mu\nu} = \bar{Q}\left[J\right]^{\lambda}{}_{\mu} \bar{t}_{\lambda\rho} \bar{Q}\left[J\right]^{\rho}{}_{\nu} . \tag{1.12}$$

This evaluates to:

$$\bar{t}\left[J\right]_{\mu\nu} = \begin{pmatrix} \bar{J}[i]^{\lambda}{}_{j} \bar{t}_{\lambda\rho} \bar{J}[i]^{\rho}{}_{k} & \bar{J}[i]^{\lambda}{}_{j} \bar{t}_{\lambda\rho} \bar{J}[o]^{\rho}{}_{o} & \bar{J}[i]^{\lambda}{}_{j} \bar{t}_{\lambda\rho} \bar{J}[a]^{\rho}{}_{\upsilon'} \\ \bar{J}[o]^{\lambda}{}_{o} \bar{t}_{\lambda\rho} \bar{J}[i]^{\rho}{}_{k} & \bar{J}[o]^{\lambda}{}_{o} \bar{t}_{\lambda\rho} \bar{J}[o]^{\rho}{}_{o} & \bar{J}[o]^{\lambda}{}_{o} \bar{t}_{\lambda\rho} \bar{J}[a]^{\rho}{}_{\upsilon'} \\ \bar{J}[a]^{\lambda}{}_{\upsilon} \bar{t}_{\lambda\rho} \bar{J}[i]^{\rho}{}_{k} & \bar{J}[a]^{\lambda}{}_{\upsilon} \bar{t}_{\lambda\rho} \bar{J}[o]^{\rho}{}_{o} & \bar{J}[a]^{\lambda}{}_{\upsilon} \bar{t}_{\lambda\rho} \bar{J}[a]^{\rho}{}_{\upsilon'} \end{pmatrix} . \tag{1.13}$$

We obtain the player-J contribution to the energy-momentum stress tensor $\bar{p}_{\mu\nu}$ from Eq. (1.10):

$$\begin{aligned} \bar{p}\left[J\right]_{\mu\nu} &= \bar{Q}\left[J\right]^{\rho}{}_{\mu} \bar{P}\left[J\right]_{\rho\sigma} \bar{Q}\left[J\right]^{\sigma}{}_{\nu} \\ \bar{P}\left[J\right]_{\rho\sigma} &\equiv \varphi_J \bar{h}_{\rho\sigma} - \eta_J \bar{\sigma}_{\rho\sigma} \end{aligned} . \tag{1.14}$$

As suggested in Vol. 1, we show below that each player is constrained by the stresses for the strategies it owns.

We draw some immediate conclusions from the general form of the stress tensor Eq. (1.11). Based on the properties of the projection operators $\bar{Q}\left[J\right]$ and it transpose $\bar{Q}\left[J\right]^{T}$, they are idempotent ($\bar{Q}\bar{Q} = \bar{Q}$) and orthogonal to the flow and each other ($\bar{Q}\left[J\right]\bar{Q}\left[K\right] = \delta_{JK}\bar{Q}\left[J\right]$), so if we project the energy momentum tensor with these operators we get:

$$\bar{Q}\left[J\right]^{\rho}{}_{\mu} \bar{T}_{\rho\sigma} \bar{Q}\left[K\right]^{\sigma}{}_{\nu} = \delta_{JK} \bar{t}\left[J\right]_{\mu\nu} . \tag{1.15}$$

In other words, the energy-momentum tensor is block-diagonal with respect to the player ownership. This supports the idea that player payoffs should have no self-payoffs (Cf. Vol. 2).

We can evaluate the pressure and shear separately, starting with the inactive-inactive contributions:

$$\begin{aligned}
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[i]^{\nu}{}_{k} &= \delta_{j}{}^{j[J]}\bar{P}[J]_{j[J]k[J]}\,\delta^{k[J]}{}_{k}\\
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[i]^{\nu}{}_{k} &= \varphi_{J}\delta_{j}{}^{j[J]}\bar{h}_{j[J]k[J]}\delta^{k[J]}{}_{k} - \eta_{J}\delta_{j}{}^{j[J]}\bar{\sigma}_{j[J]k[J]}\delta^{k[J]}{}_{k}\\
\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[i]^{\nu}{}_{k} &= \begin{cases} j,k\in o[J] & \varphi_{J}\left(\gamma_{jk}-V_{j}V_{k}\right)-\eta_{J}\bar{\sigma}_{jk}\\ otherwise & 0\end{cases}\\
J[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}J[i]^{\nu}{}_{k} &= \begin{cases} j,k\in o[J] & \varphi_{J}\left(\gamma_{jk}-V_{j}V_{k}\right)-\eta_{J}\bar{\sigma}_{jk}\\ otherwise & 0\end{cases}
\end{aligned} \quad . \quad (1.16)$$

As we expect from Eq. (1.15), only player J stresses occur for the inactive-inactive energy momentum contributions.

We expect a similar result for the active-active contributions and inactive-active contributions. For the former we have:

$$\begin{aligned}
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \delta_{\upsilon}{}^{\upsilon[J]}\bar{P}[J]_{\upsilon[J]\upsilon'[J]}\,\delta^{\upsilon'[J]}{}_{\upsilon'}\\
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \varphi_{J}\delta_{\upsilon}{}^{\upsilon[J]}\bar{h}_{\upsilon[J]\upsilon'[J]}\delta^{\upsilon'[J]}{}_{\upsilon'} - \eta_{J}\delta_{\upsilon}{}^{\upsilon[J]}\bar{\sigma}_{\upsilon[J]\upsilon'[J]}\delta^{\upsilon'[J]}{}_{\upsilon'}\\
\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon'} &= \begin{cases} \upsilon,\upsilon'\in o[J] & \varphi_{J}\left(\bar{g}_{\upsilon\upsilon'}-\bar{V}_{\upsilon}\bar{V}_{\upsilon'}\right)-\eta_{J}\bar{\sigma}_{\upsilon\upsilon'}\\ otherwise & 0\end{cases}
\end{aligned} \quad . \quad (1.17)$$

The ownership categories in the *time orthogonal frame* are the same as in the *fully co-moving frame*. We compute the contravariant forms from the above equation:

$$J[a]^{\mu\upsilon}\,\bar{p}_{\mu\nu}J[a]^{\nu\upsilon'} = \varphi_{J}\,\bar{g}^{\upsilon\upsilon[J]}\bar{h}_{\upsilon[J]\upsilon'[J]}\bar{g}^{\upsilon'[J]\upsilon'} - \eta_{J}\,\bar{g}^{\upsilon\upsilon[J]}\bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{g}^{\upsilon'[J]\upsilon'}. \quad (1.18)$$

If the transverse strategy components of the covariant metric are block-diagonal, then so will be the contravariant components and hence also the stress components of the energy momentum stress tensor.

We get for the inactive-active contributions:

$$\begin{aligned}
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon}=\delta_{j}{}^{j[J]}\bar{P}[J]_{j[J]\upsilon[J]}\,\delta^{\upsilon[J]}{}_{\upsilon}\\
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon}=-\varphi_{J}\delta_{j}{}^{j[J]}V_{j[J]}\bar{V}_{\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon}-\eta_{J}\delta_{j}{}^{j[J]}\bar{\sigma}_{j[J]\upsilon[J]}\delta^{\upsilon[J]}{}_{\upsilon}\,.\\
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[a]^{\nu}{}_{\upsilon}=\begin{cases}\mathrm{j},\upsilon\in o[J] & -\varphi_{J}V_{j}\bar{V}_{\upsilon}-\eta_{J}\bar{\sigma}_{j\upsilon}\\ otherwise & 0\end{cases}
\end{aligned}\tag{1.19}$$

From this we compute:

$$J[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}J[a]^{\nu\upsilon}=-\varphi_{J}\delta_{j}{}^{j[J]}V_{j[J]}\bar{V}_{\upsilon[J]}\bar{g}^{\upsilon[J]\upsilon}-\eta_{J}\delta_{j}{}^{j[J]}\bar{\sigma}_{j[J]\upsilon[J]}\bar{g}^{\upsilon[J]\upsilon}.\tag{1.20}$$

To the extent that the transverse strategy metric is block diagonal, the current flow for each player is determined only by the stresses for that player.

The time contributions for the inactive-time are:

$$\begin{aligned}
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o}=-\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\bar{P}[J]_{j[J]\sigma}\left(\delta^{\sigma}_{k[J]}V^{k[J]}+\delta^{\sigma}_{\upsilon[J]}\bar{V}^{\upsilon[J]}\right)\\
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o}=-\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\left(\bar{P}[J]_{j[J]k[J]}V^{k[J]}+\bar{P}[J]_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\right)\\
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o}=-\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\begin{pmatrix}\varphi_{J}h_{j[J]k[J]}V^{k[J]}-\eta_{J}\sigma_{j[J]k[J]}V^{k[J]}\\+\varphi_{J}\bar{h}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}-\eta_{J}\bar{\sigma}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\end{pmatrix}\,.\\
&\bar{J}[i]^{\mu}{}_{j}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o}=\bar{\phi}\bar{V}_{o}\delta^{j[J]}{}_{j}\begin{pmatrix}\varphi_{J}\left(V_{j[J]}\bar{\psi}_{J}-\gamma_{j[J]k[J]}V^{k[J]}\right)\\+\eta_{J}\left(\sigma_{j[J]k[J]}V^{k[J]}+\bar{\sigma}_{j[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\right)\end{pmatrix}
\end{aligned}\tag{1.21}$$

We have expressed the results in terms of the scalar:

$$\bar{\psi}_{J}\equiv V_{k[J]}V^{k[J]}+\bar{V}_{\upsilon[J]}\bar{V}^{\upsilon[J]}.\tag{1.22}$$

For the active-time contributions we have:

$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = -\delta^{\upsilon[J]}{}_{\upsilon}P[J]_{\upsilon[J]\sigma}\,\bar{\phi}\bar{V}_o\left(\delta^{\sigma}_{j[J]}V^{j[J]} + \delta^{\sigma}_{\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)$$
$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_o\delta^{\upsilon[J]}{}_{\upsilon}\left(-\bar{P}[J]_{\upsilon[J]j[J]}V^{j[J]} - \bar{P}[J]_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right). \quad (1.23)$$
$$\bar{J}[a]^{\mu}{}_{\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_o\delta^{\upsilon[J]}{}_{\upsilon}\begin{pmatrix}\varphi_J\left(\bar{V}_{\upsilon[J]}\bar{\psi}_J - \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\\ +\eta_J\left(\bar{\sigma}_{\upsilon[J]j[J]}V^{j[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\end{pmatrix}$$

We have the contravariant expression:

$$J[a]^{\mu\upsilon}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} = \bar{\phi}\bar{V}_o\bar{g}^{\upsilon\upsilon[J]}\begin{pmatrix}\varphi_J\left(\bar{V}_{\upsilon[J]}\bar{\psi}_J - \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\\ +\eta_J\left(\bar{\sigma}_{\upsilon[J]j[J]}V^{j[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]}\right)\end{pmatrix}. \quad (1.24)$$

It may be useful to note that the active and inactive strategic contributions determine the above time contributions. We can use the following argument:

$$\begin{aligned}
&\bar{p}_{\mu\nu}\bar{V}^{\nu} = 0\\
&\mu[J] \neq o \Rightarrow \bar{p}_{\mu[J]\nu[J]}\bar{V}^{\nu[J]} + \bar{p}_{\mu[J]o}\bar{V}^{o} = 0\\
&\Rightarrow \bar{h}^{\rho}{}_{\lambda[J]}\delta^{\lambda[J]}{}_{\mu[J]}\bar{P}[J]_{\rho\sigma}\bar{h}^{\sigma}{}_{\lambda[J]}\delta^{\lambda[J]}{}_{\nu[J]}\bar{V}^{\nu[J]} + \bar{p}_{\mu[J]o}\bar{V}^{o} = 0\\
&\Rightarrow \bar{\phi}\bar{V}_o\bar{h}^{\rho}{}_{\mu[J]}\bar{P}[J]_{\rho\sigma}\bar{h}^{\sigma}{}_{\nu[J]}\bar{V}^{\nu[J]} + \bar{p}_{\mu[J]o} = 0\\
&\Rightarrow \bar{p}_{\mu[J]o} = -\bar{\phi}\bar{V}_o\delta^{\rho}{}_{\mu[J]}\bar{P}[J]_{\rho\sigma}\delta^{\sigma}{}_{\nu[J]}\bar{V}^{\nu[J]}\\
&\Rightarrow \bar{p}_{\mu[J]o} = -\bar{\phi}\bar{V}_o\bar{P}[J]_{\mu[J]\nu[J]}\bar{V}^{\nu[J]}\\
&\Rightarrow \bar{p}_{\mu[J]o} = -\bar{\phi}\bar{V}_o\left(\bar{P}[J]_{\mu[J]j[J]}\bar{V}^{j[J]} + \bar{P}[J]_{\mu[J]\upsilon[J]}\bar{V}^{\upsilon[J]}\right)
\end{aligned} \quad (1.25)$$

We now expand the result:

$$\begin{aligned}
\bar{p}_{\mu[J]o} &= \bar{\phi}\bar{V}_o \begin{pmatrix} \varphi_J \left(\bar{V}_{\mu[J]}\bar{\psi}_J - \gamma_{\mu[J]k[J]}\bar{V}^{k[J]} - \bar{g}_{\mu[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]} \right) \\ +\eta_J \left(\sigma_{\mu[J]k[J]}\bar{V}^{k[J]} + \sigma_{\mu[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]} \right) \end{pmatrix} \\
\bar{p}_{j[J]o} &= \bar{\phi}\bar{V}_o \begin{pmatrix} \varphi_J \left(\bar{V}_{j[J]}\bar{\psi}_J - \gamma_{j[J]k[J]}\bar{V}^{k[J]} \right) \\ +\eta_J \sigma_{j[J]k[J]}\bar{V}^{k[J]} + \eta_J \sigma_{j[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix} \\
\bar{p}_{\upsilon[J]o} &= \bar{\phi}\bar{V}_o \begin{pmatrix} \varphi_J \left(\bar{V}_{\upsilon[J]}\bar{\psi}_J - \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]} \right) \\ +\eta_J \left(\sigma_{\upsilon[J]k[J]}\bar{V}^{k[J]} + \sigma_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon'[J]} \right) \end{pmatrix}
\end{aligned} \quad . \tag{1.26}$$

Finally, we have the time-time contributions:

$$\begin{aligned}
\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} &= \left(\bar{\phi}\bar{V}_o\right)^2 \left(\delta^{\rho}_{j[J]}V^{j[J]} + \delta^{\rho}_{\upsilon[J]}\bar{V}^{\upsilon[J]} \right) \bar{P}[J]_{\rho\sigma} \left(\delta^{\sigma}_{k[J]}V^{k[J]} + \delta^{\sigma}_{\upsilon'[J]}\bar{V}^{\upsilon'[J]} \right) \\
\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} &= \left(\bar{\phi}\bar{V}_o\right)^2 \begin{pmatrix} P[J]_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{P}[J]_{j[J]\upsilon'[J]}V^{j[J]}\bar{V}^{\upsilon'[J]} \\ +P[J]_{\upsilon[J]k[J]}V^{k[J]}\bar{V}^{\upsilon[J]} + \bar{P}[J]_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix} \\
\bar{J}[o]^{\mu}{}_{o}\,\bar{p}_{\mu\nu}\bar{J}[o]^{\nu}{}_{o} &= \left(\bar{\phi}\bar{V}_o\right)^2 \begin{pmatrix} \varphi_J \left(\gamma_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} - \bar{\psi}_J\bar{\psi}_J \right) \\ -\eta_J \begin{pmatrix} \sigma_{j[J]k[J]}V^{j[J]}V^{k[J]} + \bar{\sigma}_{j[J]\upsilon'[J]}V^{j[J]}\bar{V}^{\upsilon'[J]} \\ +\bar{\sigma}_{\upsilon[J]k[J]}V^{k[J]}\bar{V}^{\upsilon[J]} + \bar{\sigma}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix} \end{pmatrix}
\end{aligned} \quad . \tag{1.27}$$

This is also determined from the other contributions using the fact that the stress components are orthogonal to the flow:

$$\begin{aligned}
&\bar{V}^{\mu[J]}\bar{p}_{\mu[J]o} + \bar{V}^{o}\bar{p}_{oo}[J] = 0 \Rightarrow \\
&\bar{p}_{oo}[J] = \left(\bar{\phi}\bar{V}_o\right)^2 \begin{pmatrix} \varphi_J \left(\gamma_{j[J]k[J]}\bar{V}^{j[J]}\bar{V}^{k[J]} + \bar{g}_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} - \bar{\psi}_J\bar{\psi}_J \right) \\ -\eta_J \begin{pmatrix} \sigma_{j[J]k[J]}\bar{V}^{j[J]}\bar{V}^{k[J]} + \sigma_{\upsilon[J]k[J]}\bar{V}^{\upsilon[J]}\bar{V}^{k[J]} \\ +\sigma_{j[J]\upsilon'[J]}\bar{V}^{j[J]}\bar{V}^{\upsilon'[J]} + \sigma_{\upsilon[J]\upsilon'[J]}\bar{V}^{\upsilon[J]}\bar{V}^{\upsilon'[J]} \end{pmatrix} \end{pmatrix} .
\end{aligned} \tag{1.28}$$

We use these explicit expressions to interpret the energy momentum constraint contributions of the player ownership model.

The key is that the stress $\bar{p}_{\upsilon\upsilon'}$ for the active strategies, $\bar{p}_{jk}$ for the inactive strategies, and $\bar{p}_{j\upsilon}$ for the stress currents are each block-diagonal in player ownership spaces. These conditions, along with the orthogonality of flow, determine the full stress tensor.

The stresses reflect the constraints that are not explicitly included in the model. Player ownership adds stresses that in fact are specific to each player, hence justifying the analogy to immiscible fluids. Each of the fluids acts independently from the point of view of the stresses. However, there are effects that leak over since we can't assume that the metric remains block diagonal. Unless we can identify a symmetry relation, in general the metric will not be block-diagonal so there will be frame contributions that generate interactions with other players.

1.7 Acceleration in the Ownership Model

Of particular interest will be the different forces that occur in the ownership model. We know from Vol. 2 that there will in general be three distinct forces acting for the active strategies: payoff forces, cooperative forces, and inertial forces. We certainly expect the inertial forces to reflect player ownership. We may also expect changes to the other two forces. To investigate this we compute the acceleration for the inactive and active flows. As a warning, the calculations below are of necessity technical to insure that we get and can later verify that the results are correct, but they are not essential for understanding the basic idea of ownership. In the end, the important result for numerical work will be the energy momentum tensor that is obtained from these calculations.

1.7.1 *Longitudinal Conservation*

The acceleration is the covariant rate of change of the flow, which we compute from the conservation law for the energy momentum tensor. There are two sets of conservation laws: one is the longitudinal conservation law:

$$\begin{aligned}
&\bar{g}^{\lambda\rho}\bar{T}_{\mu\lambda;\rho} = \bar{g}^{\lambda\rho}\left(\mu\bar{V}_\mu\bar{V}_\lambda - \bar{p}_{\mu\lambda}\right)_{;\rho} = 0 \\
&\bar{V}^\mu\left(\mu\bar{V}_\mu\bar{V}^\lambda - \bar{p}_\mu{}^\lambda\right)_{;\lambda} = 0 \\
&\bar{V}^\lambda\mu_{;\lambda} + \mu\bar{V}^\mu\bar{V}_{\mu;\lambda}\bar{V}^\lambda + \mu\bar{V}^\lambda_{;\lambda} - \bar{V}^\mu\bar{p}_\mu{}^\lambda{}_{;\lambda} = 0 \\
&\bar{V}^\lambda\mu_{;\lambda} + \mu\bar{V}^\lambda_{;\lambda} + \bar{V}^\mu{}_{;\lambda}\bar{p}_\mu{}^\lambda = 0 \\
&\bar{V}_{\mu;\lambda}\bar{h}^{\mu\lambda} = \left(\bar{\theta}_{\mu\lambda} + \dot{\bar{V}}_\mu\bar{V}_\lambda\right)\bar{h}^{\mu\lambda} = \bar{\theta}_{\mu\lambda}\bar{h}^{\mu\lambda} = \bar{\theta}_{\mu\lambda}\bar{g}^{\mu\lambda} = \theta \\
&\bar{V}_{\mu;\lambda}\bar{p}^{\mu\lambda} = \bar{\theta}_{\mu\lambda}\bar{p}^{\mu\lambda} \\
&\bar{V}^\upsilon\partial_\upsilon\mu + \left(\mu\bar{h}_{\mu\lambda} + \bar{p}_{\mu\lambda}\right)\bar{\theta}^{\mu\lambda} = 0
\end{aligned} \quad . \qquad (1.29)$$

To evaluate this we use the results from the previous sections for the stress tensor:

$$\begin{aligned}
&\bar{V}^\upsilon\partial_\upsilon\mu + \left(\mu\bar{h}_{\mu\lambda} + \bar{p}_{\mu\lambda}\right)\bar{\theta}^{\mu\lambda} = 0 \\
&\bar{h}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = \bar{V}^\lambda_{;\lambda} = V^a{}_{|a} + g^{ab}\varpi_{bka}V^k + \gamma^{kl}\varpi_{lak}V^a \\
&\bar{h}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = V^a{}_{|a} + V^a\partial_a \ln\sqrt{\gamma} \\
&\bar{h}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = \bar{V}^a_{|a} + \bar{V}^a\partial_a \ln\sqrt{\gamma} = \partial_\upsilon\bar{V}^\upsilon + \bar{V}^\upsilon\partial_\upsilon \ln\sqrt{\bar{g}\bar{g}_{oo}\gamma} \\
&\bar{h}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = \sum_K \partial_{\upsilon[K]}\bar{V}^{\upsilon[K]} + \bar{V}^\upsilon\partial_\upsilon \ln\sqrt{\bar{g}\bar{g}_{oo}\gamma} = \bar{V}^\upsilon\partial_\upsilon \ln\sqrt{\bar{g}\bar{g}_{oo}\gamma} \\
&\sum_K \frac{1}{\sqrt{\bar{g}\bar{g}_{oo}\gamma}}\frac{\partial\left(\sqrt{\bar{g}\bar{g}_{oo}\gamma}\mu\right)}{\partial s_K} = -\bar{p}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = \bar{V}^\mu\bar{p}_\mu{}^\lambda{}_{;\lambda}
\end{aligned} \quad . \qquad (1.30)$$

1.7.2 *Transverse Conservation Law*

The other is the set of transverse conservation laws:

$$\begin{aligned}
&\bar{V}^\lambda\mu_{;\lambda}\bar{V}_\sigma + \mu\bar{V}^\lambda\bar{V}_{\sigma;\lambda} + \mu\bar{V}_\sigma\bar{\theta} - \bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu} = 0 \\
&\mu\bar{V}^\lambda\bar{V}_{\mu;\lambda} = \bar{h}_\mu{}^\sigma\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}
\end{aligned} \quad . \qquad (1.31)$$

We can approach evaluating this in two steps. First we can write the conservation laws in the *stationary holonomic frame* (where time is not orthogonal to the strategy directions):

$$\begin{aligned} \mu V^{\lambda} V_{k;\lambda} &= \mu V^{\upsilon} \partial_{\upsilon} V_k \\ \mu V^{\lambda} V_{a;\lambda} &= \mu V^{b} V_{a|b} - \mu V_k F^k_{ab} V^b - \tfrac{1}{2} \mu \partial_a \gamma_{kj} V^k V^j \end{aligned} . \tag{1.32}$$

We then transform to the *time orthogonal frame*, term by term noting that each of the terms are tensors. We start with the inactive flow:

$$\mu \bar{V}^{\lambda} \bar{V}_{k;\lambda} = \mu \bar{V}^{\upsilon} \bar{\Delta}_{\upsilon} V_k = \mu \bar{V}^{\upsilon} \left(\partial_{\upsilon} - a_{\upsilon} \partial_o \right) V_k = \mu \bar{V}^{\upsilon} \partial_{\upsilon} V_k . \tag{1.33}$$

This is unchanged, as we should expect since the inactive directions are not transformed. Next we look at the active components:

$$\mu \bar{V}^{\lambda} \bar{V}_{a;\lambda} = \mu \bar{V}^{b} \bar{V}_{a|b} - \mu V_k \bar{F}^k_{ab} \bar{V}^b - \tfrac{1}{2} \mu \bar{\Delta}_a \gamma_{kj} V^k V^j . \tag{1.34}$$

From this we pull out the time component:

$$\begin{aligned} \mu \bar{V}^{\lambda} \bar{V}_{o;\lambda} &= \mu \bar{V}^{b} \bar{V}_{o|b} - \mu V_k \bar{F}^k_{o\upsilon} \bar{V}^{\upsilon} \\ \mu \bar{V}^{\lambda} \bar{V}_{o;\lambda} &= \mu \bar{V}^{\upsilon} \partial_{\upsilon} \bar{V}_o - \mu V_k \bar{F}^k_{o\upsilon} \bar{V}^{\upsilon} \end{aligned} . \tag{1.35}$$

Finally, we compute the strategy components:

$$\begin{aligned} \mu \bar{V}^{\lambda} \bar{V}_{\upsilon;\lambda} &= \mu \bar{V}^{b} \bar{V}_{\upsilon|b} - \mu V_k \bar{F}^k_{\upsilon b} \bar{V}^b - \tfrac{1}{2} \mu \bar{\Delta}_{\upsilon} \gamma_{kj} V^k V^j \\ \mu \bar{V}^{\lambda} \bar{V}_{\upsilon;\lambda} &= \begin{pmatrix} \mu \bar{V}^{\upsilon'} \partial_{\upsilon'} \bar{V}_{\upsilon} - \tfrac{1}{2} \mu \bar{V}^{\upsilon'} \bar{V}^{\upsilon''} \partial_{\upsilon} \bar{g}_{\upsilon''\upsilon'} \\ -\tfrac{1}{2} \mu \bar{V}^{o} \bar{V}^{o} \partial_{\upsilon} \bar{g}_{oo} - \tfrac{1}{2} \mu V^k V^j \partial_{\upsilon} \gamma_{kj} \\ -\mu \bar{V}_o \left(\partial_{\upsilon} a_{\upsilon'} - \partial_{\upsilon'} a_{\upsilon} \right) \bar{V}^{\upsilon'} - \mu V_k \left(\bar{F}^k_{\upsilon o} \bar{V}^o + \bar{F}^k_{\upsilon\upsilon'} \bar{V}^{\upsilon'} \right) \end{pmatrix} . \end{aligned} \tag{1.36}$$

1.7.3 *Transverse Equations in terms of Stresses*

We have computed the accelerations; now we compute the stress contributions. We obtain the transverse acceleration Eq. (1.31) for each set:

$$
\begin{aligned}
&\bar{V}^{\sigma}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}=-\bar{\theta}_{\mu\lambda}\bar{p}^{\mu\lambda}\\
&\bar{V}^{\sigma}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}=\bar{V}^{j}\bar{p}_{j\nu;\lambda}\bar{g}^{\lambda\nu}+\bar{V}^{o}\bar{p}_{o\nu;\lambda}\bar{g}^{\lambda\nu}+\bar{V}^{\upsilon}\bar{p}_{\upsilon\nu;\lambda}\bar{g}^{\lambda\nu}\\
&\mu\dot{\bar{V}}_{j[J]}=\bar{h}^{\sigma}{}_{j[J]}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}=\bar{p}_{j[J]\nu;\lambda}\bar{g}^{\lambda\nu}-\bar{V}_{j[J]}\bar{V}^{\sigma}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}\ .\\
&\mu\dot{\bar{V}}_{\upsilon[J]}=\bar{h}^{\sigma}{}_{\upsilon[J]}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}=\bar{p}_{\upsilon[J]\nu;\lambda}\bar{g}^{\lambda\nu}-\bar{V}_{\upsilon[J]}\bar{V}^{\sigma}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}\\
&\mu\dot{\bar{V}}_{o}=\bar{h}^{\sigma}{}_{o}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}=\bar{p}_{o\nu;\lambda}\bar{g}^{\lambda\nu}-\bar{V}_{o}\bar{V}^{\sigma}\bar{p}_{\sigma\nu;\lambda}\bar{g}^{\lambda\nu}
\end{aligned}
\tag{1.37}
$$

From this we need to compute the stress contributions. We proceed as in the previous section.

We start with the first term of the inactive contribution in the holonomic frame:

$$
\begin{aligned}
&p_{j\nu;\lambda}g^{\lambda\nu}=\left(\partial_{\lambda}p_{j\nu}-\varpi^{\mu}{}_{j\lambda}p_{\mu\nu}-\varpi^{\mu}{}_{\nu\lambda}p_{j\mu}\right)g^{\lambda\nu}\\
&p_{j\nu;\lambda}g^{\lambda\nu}=\begin{pmatrix}\left(\partial_{k}p_{jl}-\varpi^{\mu}{}_{jk}p_{\mu l}-\varpi^{\mu}{}_{lk}p_{j\mu}\right)g^{kl}\\+\left(\partial_{a}p_{jb}-\varpi^{\mu}{}_{ja}p_{\mu b}-\varpi^{\mu}{}_{ba}p_{j\mu}\right)g^{ab}\end{pmatrix}.\\
&p_{j\nu;\lambda}g^{\lambda\nu}=\begin{pmatrix}\tfrac{1}{2}\partial_{a}\gamma_{jk}p^{ak}+\tfrac{1}{2}\gamma^{kl}\partial_{a}\gamma_{lk}p^{a}{}_{j}\\+g^{ab}p_{jb|a}-\tfrac{1}{2}\partial_{a}\gamma_{jk}p^{ka}\end{pmatrix}\\
&p_{j\nu;\lambda}g^{\lambda\nu}=g^{ab}p_{jb|a}+\tfrac{1}{2}\gamma^{kl}\partial_{a}\gamma_{lk}p^{a}{}_{j}
\end{aligned}
\tag{1.38}
$$

This is a tensor expression so it can be expressed in the *time orthogonal frame* as well:

$$\begin{aligned}
&\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} = \left(g^{ab} p_{jb|a} + \tfrac{1}{2}\gamma^{kl}\partial_a \gamma_{lk} p^a{}_j \right) \\
&\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} = \begin{pmatrix} \overline{g}^{\upsilon\upsilon'}\partial_\upsilon \overline{p}_{j\upsilon'} - \overline{g}^{oo}\overline{p}^{\upsilon}_j \varpi_{\upsilon oo} - \overline{g}^{\upsilon\upsilon'}\overline{p}^{\upsilon''}_j \varpi_{\upsilon''\upsilon'\upsilon} \\ + \tfrac{1}{2}\gamma^{kl}\partial_\upsilon \gamma_{lk}\overline{p}^{\upsilon}_j \end{pmatrix} \\
&\tilde{g} = \overline{g}\overline{g}_{oo} \\
&\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} = \begin{pmatrix} \overline{g}^{\upsilon\upsilon'}\partial_\upsilon \overline{p}_{j\upsilon'} + \tfrac{1}{2}\overline{g}^{oo}\partial_\upsilon \overline{g}_{oo}\overline{p}^{\upsilon}_j + \tfrac{1}{2}\overline{g}^{\upsilon\upsilon'}\partial_{\upsilon''}\overline{g}_{\upsilon'\upsilon}\overline{p}^{\upsilon''} \\ -\overline{g}^{\upsilon\upsilon'}\partial_{\upsilon'}\overline{g}_{\upsilon''\upsilon}\overline{p}^{\upsilon''}_j + \tfrac{1}{2}\gamma^{kl}\partial_\upsilon \gamma_{lk}\overline{p}^{\upsilon}_j \end{pmatrix} \\
&\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} = \overline{g}^{\upsilon\upsilon'}\partial_\upsilon \overline{p}_{j\upsilon'} + \partial_\upsilon \ln\sqrt{\overline{g}\overline{g}_{oo}\gamma}\,\overline{p}^{\upsilon}_j - \overline{g}^{\upsilon\upsilon'}\partial_\upsilon \overline{g}_{\upsilon''\upsilon'}\overline{p}^{\upsilon''}_j \\
&\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} = \frac{1}{\sqrt{\overline{g}\overline{g}_{oo}\gamma}}\partial_\upsilon \left(\sqrt{\overline{g}\overline{g}_{oo}\gamma}\,\overline{g}^{\upsilon\upsilon'}\overline{p}_{j\upsilon'} \right)
\end{aligned} \quad . \tag{1.39}$$

In similar fashion we compute the transverse strategy contributions, starting with the expression in the *stationary holonomic frame*:

$$\begin{aligned}
&p_{a\nu;\lambda} g^{\lambda\nu} = \left(\partial_\lambda p_{a\nu} - \varpi_{\mu a\lambda} p^\mu{}_\nu - \varpi_{\mu\nu\lambda} p^\mu{}_a \right) g^{\lambda\nu} \\
&p_{a\nu;\lambda} g^{\lambda\nu} = \begin{pmatrix} \left(\partial_k p_{aj} - \varpi_{\mu ak} p^\mu{}_j - \varpi_{\mu jk} p^\mu{}_a \right) \gamma^{jk} \\ + \left(\partial_c p_{ab} - \varpi_{\mu ac} p^\mu{}_b - \varpi_{\mu bc} p^\mu{}_a \right) g^{bc} \end{pmatrix} \\
&p_{a\nu;\lambda} g^{\lambda\nu} = \begin{pmatrix} g^{bc} p_{ab|c} - \varpi_{jac} p^{jc} - g^{bc}\varpi_{jbc} p^j{}_a \\ -\varpi_{bak} p^{bk} - \gamma^{jk}\varpi_{bjk} p^b{}_a - \varpi_{lak} p^{lk} \end{pmatrix} \\
&p_{a\nu;\lambda} g^{\lambda\nu} = g^{bc} p_{ab|c} - F^k_{ab} p_k{}^b + \tfrac{1}{2} p_a{}^b \partial_b \ln\gamma - \tfrac{1}{2} p^{lk}\partial_a \gamma_{kl}
\end{aligned} \quad . \tag{1.40}$$

We first take the expression for the transverse strategies now expressed in the *time orthogonal frame*:

$$
\begin{aligned}
\overline{p}_{a\nu;\lambda}\overline{g}^{\lambda\nu} &= \overline{g}^{bc}\overline{p}_{ab|c} - \overline{F}^{k}{}_{ab}\overline{p}_{k}{}^{b} + \tfrac{1}{2}\overline{p}_{a}{}^{b}\partial_{b}\ln\gamma - \tfrac{1}{2}p^{lk}\partial_{a}\gamma_{kl} \\
\overline{p}_{\upsilon\nu;\lambda}\overline{g}^{\lambda\nu} &= \overline{g}^{bc}\overline{p}_{\upsilon b|c} - \overline{F}^{k}{}_{\upsilon b}\overline{p}_{k}{}^{b} + \tfrac{1}{2}\overline{p}_{\upsilon}{}^{b}\partial_{b}\ln\gamma - \tfrac{1}{2}p^{lk}\partial_{\upsilon}\gamma_{kl} \\
\overline{p}_{\upsilon\nu;\lambda}\overline{g}^{\lambda\nu} &= \begin{pmatrix}
\overline{g}^{\upsilon''\upsilon'}\partial_{\upsilon''}\overline{p}_{\upsilon\upsilon'} - \overline{p}^{\upsilon''\upsilon'''}\varpi_{\upsilon'''\upsilon\upsilon''} - \overline{g}^{\upsilon'\upsilon''}\overline{p}_{\upsilon}{}^{\upsilon'''}\varpi_{\upsilon'''\upsilon'\upsilon''} \\
-\overline{g}^{\upsilon'\upsilon''}\overline{p}_{\upsilon'}{}^{o}\varpi_{o\upsilon\upsilon''} - \overline{g}^{\upsilon'\upsilon''}\overline{p}_{\upsilon}{}^{o}\varpi_{o\upsilon'\upsilon''} \\
-\tfrac{1}{2}\overline{p}_{o}{}^{\upsilon'}\left(\partial_{\upsilon}a_{\upsilon'} - \partial_{\upsilon'}a_{\upsilon}\right) - \tfrac{1}{2}\overline{p}_{o}{}^{o}\partial_{\upsilon}\ln\overline{g}_{oo} + \tfrac{1}{2}\overline{p}_{\upsilon}{}^{\upsilon'}\partial_{\upsilon}\ln\overline{g}_{oo} \\
-\overline{F}^{k}{}_{\upsilon b}\overline{p}_{k}{}^{b} + \tfrac{1}{2}\overline{p}_{\upsilon}{}^{b}\partial_{b}\ln\gamma - \tfrac{1}{2}p^{lk}\partial_{\upsilon}\gamma_{kl}
\end{pmatrix} \\
\overline{p}_{\upsilon\nu;\lambda}\overline{g}^{\lambda\nu} &= \begin{pmatrix}
\dfrac{1}{\sqrt{\overline{g}\overline{g}_{oo}\gamma}}\partial_{\upsilon'}\left(\sqrt{\overline{g}\overline{g}_{oo}\gamma}\,\overline{g}^{\upsilon'\upsilon''}\overline{p}_{\upsilon\upsilon''}\right) \\
-\tfrac{1}{2}\overline{p}^{\upsilon''\upsilon'''}\partial_{\upsilon}\overline{g}_{\upsilon''\upsilon'''} - \tfrac{1}{2}\overline{p}^{oo}\partial_{\upsilon}\overline{g}_{oo} - \tfrac{1}{2}p^{lk}\partial_{\upsilon}\gamma_{kl} \\
-\overline{p}_{o}{}^{\upsilon'}\left(\partial_{\upsilon}a_{\upsilon'} - \partial_{\upsilon'}a_{\upsilon}\right) - \overline{F}^{k}{}_{\upsilon o}\overline{p}_{k}{}^{o} - \overline{F}^{k}{}_{\upsilon\upsilon'}\overline{p}_{k}{}^{\upsilon'} \\
\\
\end{pmatrix}
\end{aligned}
\tag{1.41}
$$

Finally there is the time component, which we now compute in the *time orthogonal frame*:

$$
\begin{aligned}
\overline{p}_{o\nu;\lambda}\overline{g}^{\lambda\nu} &= \overline{g}^{bc}\overline{p}_{ob|c} - \overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon} + \tfrac{1}{2}\overline{p}_{o}{}^{\upsilon}\partial_{\upsilon}\ln\gamma \\
\overline{p}_{o\nu;\lambda}\overline{g}^{\lambda\nu} &= \begin{pmatrix}
\overline{g}^{bc}\partial_{c}\overline{p}_{ob} - \overline{g}^{bc}\overline{p}_{b}{}^{d}\varpi_{doc} - \overline{g}^{bc}\overline{p}_{o}{}^{d}\varpi_{dbc} \\
-\overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon} + \tfrac{1}{2}\overline{p}_{o}{}^{\upsilon}\partial_{\upsilon}\ln\gamma
\end{pmatrix} \\
\overline{p}_{o\nu;\lambda}\overline{g}^{\lambda\nu} &= \begin{pmatrix}
\overline{g}^{bc}\partial_{c}\overline{p}_{ob} - \overline{g}^{\upsilon\upsilon'}\overline{p}_{\upsilon}{}^{\upsilon''}\varpi_{\upsilon''o\upsilon'} - \overline{g}^{\upsilon\upsilon'}\overline{p}_{o}{}^{\upsilon''}\varpi_{\upsilon''\upsilon\upsilon'} \\
-\overline{g}^{\upsilon\upsilon'}\overline{p}_{\upsilon}{}^{o}\varpi_{oo\upsilon'} - \overline{g}^{\upsilon\upsilon'}\overline{p}_{o}{}^{o}\varpi_{o\upsilon\upsilon'} \\
-2\overline{g}^{oo}\overline{p}_{o}{}^{\upsilon}\varpi_{\upsilon oo} \\
-\overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon} + \tfrac{1}{2}\overline{p}_{o}{}^{\upsilon}\partial_{\upsilon}\ln\gamma
\end{pmatrix}. \\
\overline{p}_{o\nu;\lambda}\overline{g}^{\lambda\nu} &= \frac{1}{\sqrt{\overline{g}\overline{g}_{oo}\gamma}}\partial_{\upsilon}\left(\sqrt{\overline{g}\overline{g}_{oo}\gamma}\,\overline{g}^{\upsilon\upsilon'}\overline{p}_{o\upsilon'}\right) - \overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon}
\end{aligned}
\tag{1.42}
$$

1.7.4 *Equations in the Covariant Gauge*

In the covariant gauge, these expressions in the *time orthogonal frame* simplify:

$$\begin{aligned}
\overline{p}_{j\nu;\lambda}\overline{g}^{\lambda\nu} &= \overline{g}^{\upsilon\upsilon'}\partial_{\upsilon}\overline{p}_{j\upsilon'} \\
\overline{p}_{o\nu;\lambda}\overline{g}^{\lambda\nu} &= \overline{g}^{\upsilon\upsilon'}\partial_{\upsilon}\overline{p}_{o\upsilon'} - \overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon} \\
\overline{p}_{\upsilon\nu;\lambda}\overline{g}^{\lambda\nu} &= \begin{pmatrix} \overline{g}^{\upsilon'\upsilon''}\partial_{\upsilon'}\overline{p}_{\upsilon\upsilon''} \\ -\tfrac{1}{2}\overline{p}^{\upsilon''\upsilon'''}\partial_{\upsilon}\overline{g}_{\upsilon''\upsilon'''} - \tfrac{1}{2}\overline{p}^{oo}\partial_{\upsilon}\overline{g}_{oo} - \tfrac{1}{2}p^{jk}\partial_{\upsilon}\gamma_{jk} \\ -\overline{p}_{o}{}^{\upsilon'}\left(\partial_{\upsilon}a_{\upsilon'} - \partial_{\upsilon'}a_{\upsilon}\right) - \overline{F}^{k}{}_{\upsilon o}\overline{p}_{k}{}^{o} - \overline{F}^{k}{}_{\upsilon\upsilon'}\overline{p}_{k}{}^{\upsilon'} \end{pmatrix}
\end{aligned} \quad . \quad (1.43)$$

1.7.5 *Longitudinal Equations*

Given these expressions we can project the flow onto them to get the contribution to the longitudinal conservation law:

$$V^{\mu}p_{\mu\nu;\lambda}g^{\nu\lambda} = \begin{pmatrix} \overline{g}^{\upsilon\upsilon'}\overline{V}^{j}\partial_{\upsilon}\overline{p}_{j\upsilon'} + \overline{g}^{\upsilon\upsilon'}\overline{V}^{o}\partial_{\upsilon}\overline{p}_{o\upsilon'} + \overline{g}^{\upsilon'\upsilon''}\overline{V}^{\upsilon}\partial_{\upsilon'}\overline{p}_{\upsilon\upsilon''} \\ -\overline{V}^{o}\overline{F}^{k}{}_{o\upsilon}\overline{p}_{k}{}^{\upsilon} - \overline{V}^{\upsilon}\overline{F}^{k}{}_{\upsilon o}\overline{p}_{k}{}^{o} - \overline{F}^{k}{}_{\upsilon\upsilon'}\overline{V}^{\upsilon}\overline{p}_{k}{}^{\upsilon'} \\ -\overline{p}_{o}{}^{\upsilon'}\left(\partial_{\upsilon}a_{\upsilon'} - \partial_{\upsilon'}a_{\upsilon}\right)\overline{V}^{\upsilon} \\ -\tfrac{1}{2}\overline{p}^{\upsilon''\upsilon'''}\overline{V}^{\upsilon}\partial_{\upsilon}\overline{g}_{\upsilon''\upsilon'''} - \tfrac{1}{2}\overline{p}^{oo}\overline{V}^{\upsilon}\partial_{\upsilon}\overline{g}_{oo} - \tfrac{1}{2}p^{jk}\overline{V}^{\upsilon}\partial_{\upsilon}\gamma_{jk} \end{pmatrix}. \quad (1.44)$$

1.7.6 *Ownership Equations for Inactive Flow*

By following these technical steps and working in the covariant gauge, we achieve our desired results, starting with an expression for the inactive flow with the ownership attributes identified (*Cf.* Eq. (1.37)):

$$\mu\overline{V}^{\upsilon}\partial_{\upsilon}V_{j[J]} = \overline{g}^{\upsilon\upsilon[J]}\partial_{\upsilon}\overline{P}\left[J\right]_{j[J]\upsilon[J]} + V_{j[J]}\overline{\theta}_{\nu\lambda}\overline{p}^{\nu\lambda}. \quad (1.45)$$

1.7.7 *Ownership Equations for Active Transverse Flow*

The active flow equations are similar:

$$\mu\overline{g}_{\upsilon[J]\upsilon'}\dot{V}^{\upsilon'} = \overline{p}_{\upsilon[J]\nu;\lambda}\overline{g}^{\lambda\nu} + \overline{V}_{\upsilon[J]}\overline{\theta}_{\nu\lambda}\overline{p}^{\nu\lambda}. \quad (1.46)$$

We expand this expression somewhat and collect terms:

$$\mu\bar{V}^{\upsilon'}\partial_{\upsilon'}\bar{V}_{\upsilon}=\begin{pmatrix}\tfrac{1}{2}\mu\bar{V}^{\upsilon'}\bar{V}^{\upsilon''}\partial_{\upsilon}\bar{g}_{\upsilon''\upsilon'}\\+\tfrac{1}{2}\mu\bar{V}^{o}\bar{V}^{o}\partial_{\upsilon}\bar{g}_{oo}+\tfrac{1}{2}\mu V^{k}V^{j}\partial_{\upsilon}\gamma_{kj}\\+\mu\bar{V}_{o}\left(\partial_{\upsilon}a_{\upsilon'}-\partial_{\upsilon'}a_{\upsilon}\right)\bar{V}^{\upsilon'}+\mu V_{k}\left(\bar{F}^{k}_{\upsilon o}\bar{V}^{o}+\bar{F}^{k}_{\upsilon\upsilon'}\bar{V}^{\upsilon'}\right)\\+\bar{g}^{\upsilon'\upsilon''}\partial_{\upsilon'}\bar{p}_{\upsilon\upsilon''}+\bar{V}_{\upsilon}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}\\-\tfrac{1}{2}\bar{p}^{\upsilon''\upsilon'''}\partial_{\upsilon}\bar{g}_{\upsilon''\upsilon'''}-\tfrac{1}{2}\bar{p}^{oo}\partial_{\upsilon}\bar{g}_{oo}-\tfrac{1}{2}p^{jk}\partial_{\upsilon}\gamma_{jk}\\-\bar{p}_{o}{}^{\upsilon'}\left(\partial_{\upsilon}a_{\upsilon'}-\partial_{\upsilon'}a_{\upsilon}\right)-\bar{F}^{k}{}_{\upsilon o}\bar{p}_{k}{}^{o}-\bar{F}^{k}{}_{\upsilon\upsilon'}\bar{p}_{k}{}^{\upsilon'}\end{pmatrix}. \quad (1.47)$$

We collect the terms as follows:

$$\mu\bar{V}^{\upsilon'}\partial_{\upsilon'}\bar{V}_{\upsilon[J]}=\bar{g}^{\upsilon'\upsilon''[J]}\partial_{\upsilon'}\bar{p}_{\upsilon[J]\upsilon''[J]}+\mathcal{C}_{\upsilon[J]}+\mathcal{R}_{\upsilon[J]}. \quad (1.48)$$

We have introduced the following for the new cooperative and payoff terms:

$$\bar{f}_{\upsilon\upsilon'}\equiv\partial_{\upsilon}a_{\upsilon'}-\partial_{\upsilon'}a_{\upsilon}$$

$$\mathcal{C}_{\upsilon[J]}\equiv\begin{pmatrix}\tfrac{1}{2}\left(\mu\bar{V}^{\upsilon'}\bar{V}^{\upsilon''}-\bar{p}^{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}_{\upsilon''\upsilon'}+\bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}\\+\tfrac{1}{2}\left(\mu\bar{V}^{o}\bar{V}^{o}-\bar{p}^{oo}\right)\partial_{\upsilon[J]}\bar{g}_{oo}+\tfrac{1}{2}\left(\mu V^{k}V^{j}-p^{jk}\right)\partial_{\upsilon[J]}\gamma_{jk}\end{pmatrix}. \quad (1.49)$$

$$\mathcal{R}_{\upsilon[J]}\equiv\begin{pmatrix}\bar{F}^{k}_{\upsilon[J]\upsilon'}\left(\mu V_{k}\bar{V}^{\upsilon'}-\bar{p}_{k}{}^{\upsilon'}\right)\\+\bar{f}_{\upsilon[J]\upsilon'}\left(\mu\bar{V}_{o}\bar{V}^{\upsilon'}-\bar{p}_{o}{}^{\upsilon'}\right)+\bar{F}^{k}_{\upsilon[J]o}\left(\mu V_{k}\bar{V}^{o}-\bar{p}_{k}{}^{o}\right)\end{pmatrix}$$

1.7.8 *Ownership Equations for the Time Flow*

In a similar fashion we compute the time flow equations in the *time orthogonal frame*:

$$\mu\bar{V}^{\upsilon}\partial_{\upsilon}\bar{V}_{o}=\bar{g}^{\upsilon\upsilon'}\partial_{\upsilon}\bar{p}_{o\upsilon'}+\bar{F}^{k}{}_{o\upsilon}\left(\mu V_{k}\bar{V}^{\upsilon}-\bar{p}_{k}{}^{\upsilon}\right)+\bar{V}_{o}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}. \quad (1.50)$$

1.8 Ownership Streamlines

In analogy to the concept of streamlines for fluids, which are lines along the flow, we introduce ***ownership streamlines***, which are lines along the player effort directions. For ordinary streamlines, it is often convenient to assume that the flow has no compression, $\bar{g}^{\upsilon\upsilon'}\partial_\upsilon \bar{V}_{\upsilon'} = 0$. We follow a similar path but assume that this is true for each player subspace:

- For each player J we have

$$\bar{g}^{\upsilon\upsilon[J]}\partial_\upsilon \bar{V}_{\upsilon[J]} = 0 ; \tag{1.51}$$

- The metric is ***player block diagonal***

$$\bar{g}^{\upsilon[J]\upsilon'[K]} = \delta^{JK}\bar{g}^{\upsilon[J]\upsilon'[J]} ; \tag{1.52}$$

- For practical calculations, we assume that viscosity term is absent so that we compute everything in terms of the ***player pressure*** scalar φ_J.

In analogy to the ideal fluid, this is the ***ideal ownership model***. We intend to use this model for numerical calculations. We can use this model with any assumption on the number of players and any assumption on the values for the player payoff matrix $\bar{F}_{\upsilon\upsilon'}{}^k$.

1.8.1 *Player Streamlines*

For the *ideal fluid*, there is a single streamline and it follows the coordinate curve of the energy flow. For the *ideal ownership model*, we have an active strategy coordinate flow for each player, defined as

$$\mathcal{S}_{\mathbf{J}}{}^{\upsilon} \equiv \bar{g}^{\upsilon\upsilon[J]}\bar{V}_{\upsilon[J]} . \tag{1.53}$$

The coordinate curve for this follows the corresponding flow for the indicated player.

The active strategy flow $\bar{V}^\upsilon$ can be decomposed into these flows:

$$\bar{V}^\upsilon = \sum_J \bar{g}^{\upsilon\upsilon'[J]}\bar{V}_{\upsilon'[J]} = \sum_J \mathcal{S}_{\mathbf{J}}{}^{\upsilon} . \tag{1.54}$$

Based on these ***player streamlines***, there will be coordinate curves defined with a parameter s_J for each player:

$$\frac{\partial}{\partial s_J} \equiv \mathcal{S}_{\mathbf{J}}{}^{\upsilon}\partial_{\upsilon}\,. \tag{1.55}$$

In general, these vectors don't commute: $\left[\mathcal{S}_{\mathbf{J}},\ \mathcal{S}_{\mathbf{K}}\right] \neq 0$. This suggests that we can only define preferences for each player:

$$\frac{\partial y^{\upsilon[J]}}{\partial s_J} = \mathcal{S}_{\mathbf{J}}{}^{\upsilon[J]}\,. \tag{1.56}$$

1.8.2 *Inactive Flow Equation for Ideal Ownership Model*

We are now in a position to get a more transparent form for the transverse and longitudinal equations. We compute the inactive flow equation from Eq. (1.45):

$$\begin{aligned}
&\mu\bar{V}^{\upsilon}\partial_{\upsilon}V_{j[J]} = -\bar{g}^{\upsilon[J]\upsilon'[J]}\bar{V}_{\upsilon'[J]}\partial_{\upsilon[J]}\left(\varphi_J\bar{V}_{j[J]}\right) + V_{j[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\
&\mu\sum_K \mathcal{S}_{\mathbf{K}}{}^{\upsilon}\partial_{\upsilon}V_{j[J]} = -\mathcal{S}_{\mathbf{J}}{}^{\upsilon[J]}\partial_{\upsilon[J]}\left(\varphi_J\bar{V}_{j[J]}\right) + V_{j[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\
&\mu\sum_K \partial_{s_K}V_{j[J]} = -\partial_{s_J}\left(\varphi_J\bar{V}_{j[J]}\right) + V_{j[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\
&\left(\mu+\varphi_J\right)\frac{\partial V_{j[J]}}{\partial s_J} + \sum_{K\neq J}\mu\frac{\partial V_{j[J]}}{\partial s_K} = -V_{j[J]}\frac{\partial \varphi_J}{\partial s_J} + V_{j[J]}\theta_{\nu\lambda}p^{\nu\lambda}
\end{aligned}\quad . \tag{1.57}$$

We have a partial differential equation for the coordinate curves of the inactive flow given a choice of stresses for each of the players. We will provide an expression for $\theta_{\nu\lambda}p^{\nu\lambda}$ below. The distinctively new feature is the need to consider multiple coordinate curves. We have a ***mesh-line*** solution as opposed to a streamline solution. For a given player-J, there are different weights depending on the actual values of the player stresses.

1.8.3 *Time Flow Equation*

The time flow equations follow from Eq. (1.50):

$$\mu \bar{V}^{\upsilon}\partial_{\upsilon}\bar{V}_{o} = \bar{g}^{\upsilon\upsilon'}\partial_{\upsilon}\bar{p}_{o\upsilon'} + \bar{F}^{k}{}_{o\upsilon}\left(\mu V_{k}\bar{V}^{\upsilon} - \bar{p}_{k}{}^{\upsilon}\right) + \bar{V}_{o}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}$$

$$\begin{pmatrix} \sum_{K}\left(\mu + \varphi_{K}\bar{\phi}\left(1-\bar{\psi}_{K}\right)\right)\dfrac{\partial \bar{V}_{o}}{\partial s_{K}} \\ +\bar{V}_{o}\sum_{K}\dfrac{\partial\left(\varphi_{K}\bar{\phi}\left(1-\bar{\psi}_{K}\right)\right)}{\partial s_{K}} - \bar{V}_{o}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\ -\sum_{K,L}\bar{F}^{k[K]}{}_{o\upsilon[L]}\left(\mu V_{k[K]}\bar{V}^{\upsilon[L]} + \delta_{KL}\varphi_{K}\bar{V}_{k[K]}\bar{V}^{\upsilon[K]}\right) \end{pmatrix} = 0 \quad . \qquad (1.58)$$

1.8.4 *Active Flow Equations*

The behaviors for the active flow are determined from Eq. (1.48):

$$\sum_{K}\mu\frac{\partial V_{\upsilon[J]}}{\partial s_{K}} + \varphi_{J}\frac{\partial \bar{V}_{\upsilon[J]}}{\partial s_{J}} = \begin{pmatrix} \varphi_{J}\bar{g}^{\upsilon'\upsilon''[J]}\partial_{\upsilon'}\bar{g}_{\upsilon[J]\upsilon''[J]} + \partial_{\upsilon[J]}\varphi_{J} \\ -\bar{V}_{\upsilon[J]}\bar{V}^{\upsilon'[J]}\partial_{\upsilon'[J]}\varphi_{J} + \bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\ -\tfrac{1}{2}\left(\mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \bar{p}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} \\ -\tfrac{1}{2}\left(\mu V_{k}V_{j} - p_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk} \\ -\tfrac{1}{2}\left(\mu\bar{V}_{o}\bar{V}_{o} - \bar{p}_{oo}\right)\partial_{\upsilon[J]}\bar{g}^{oo} \\ +\bar{f}_{\upsilon[J]\upsilon'}\left(\mu\bar{V}_{o}\bar{V}^{\upsilon'} - \bar{p}_{o}{}^{\upsilon'}\right) \\ +\bar{F}^{k}_{\upsilon[J]\upsilon'}\left(\mu V_{k}\bar{V}^{\upsilon'} - \bar{p}_{k}{}^{\upsilon'}\right) \\ +\bar{F}^{k}_{\upsilon[J]o}\left(\mu V_{k}\bar{V}^{o} - \bar{p}_{k}{}^{o}\right) \end{pmatrix} . \qquad (1.59)$$

The evaluation of the scalars such as the pressure and metric components, which are given functions of the preference coordinates, uses the preferences defined in Eq. (1.56). The time components of the pressure (Eqs. (1.26) and (1.28)) simplify based on our assumptions at the start of the section:

$$\begin{aligned}
&\bar{p}_{j[J]}{}^{o} = -\bar{\phi}\bar{V}^{o}\varphi_J\bar{V}_{j[J]}\left(1-\bar{\psi}_J\right) \Rightarrow \bar{p}_{o}{}^{j[J]} = -\bar{\phi}\bar{V}_{o}\varphi_J\bar{V}^{j[J]}\left(1-\bar{\psi}_J\right)\\
&\bar{p}_{\upsilon[J]}{}^{o} = -\bar{\phi}\bar{V}^{o}\varphi_J\bar{V}_{\upsilon[J]}\left(1-\bar{\psi}_J\right) \Rightarrow \bar{p}_{o}{}^{\upsilon[J]} = -\bar{\phi}\bar{V}_{o}\varphi_J\bar{V}^{\upsilon[J]}\left(1-\bar{\psi}_J\right).\\
&\bar{p}_{oo}\left[J\right] = \bar{\phi}^{2}\bar{V}_{o}\bar{V}_{o}\varphi_J\bar{\psi}_J\left(1-\bar{\psi}_J\right) \Rightarrow \bar{p}_{o}{}^{o}\left[J\right] = \bar{\phi}\varphi_J\bar{\psi}_J\left(1-\bar{\psi}_J\right)
\end{aligned} \tag{1.60}$$

We obtain the following result for the active flow:

$$\sum_K \mu\frac{\partial V_{\upsilon[J]}}{\partial s_K} + \varphi_J\frac{\partial \bar{V}_{\upsilon[J]}}{\partial s_J} = \begin{pmatrix}
\partial_{\upsilon[J]}\varphi_J - \bar{V}_{\upsilon[J]}\bar{V}^{\upsilon'[J]}\partial_{\upsilon'[J]}\varphi_J + \bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}\\
+\varphi_J\bar{g}^{\upsilon'[J]\upsilon''[J]}\partial_{\upsilon'[J]}\bar{g}_{\upsilon[J]\upsilon''[J]} - \tfrac{1}{2}\left(\mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \bar{p}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'}\\
-\tfrac{1}{2}\left(\mu\bar{V}_{o}\bar{V}_{o} - \bar{p}_{oo}\right)\partial_{\upsilon[J]}\bar{g}^{oo} - \tfrac{1}{2}\left(\mu V_k V_j - p_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk}\\
+\sum_K \bar{V}_{o}\bar{\phi}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)\bar{f}_{\upsilon[J]\upsilon'[K]}\mathcal{S}_{\mathbf{K}}{}^{\upsilon'[K]}\\
+\sum_K\left(\varphi_K\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'[K]} + \sum_L \mu V_{l[L]}\bar{F}^{l[L]}{}_{\upsilon[J]\upsilon'[K]}\right)\mathcal{S}_{\mathbf{K}}{}^{\upsilon'[K]}\\
-\sum_K \bar{\phi}\bar{V}^{o}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]}
\end{pmatrix}. \tag{1.61}$$

The ownership forces are clearly displayed. The key elements are analogous but not identical to the ideal fluid. The acceleration along each mesh line is determined by an inertial force based on the gradient of the stress as well as by payoff flows which reflect the character of how the players interact. For later use, we have included the metric behaviors as well.

1.8.5 *Longitudinal Equation for Ideal Ownership Model*

The remaining equation is the longitudinal conservation law. We use Eq. (1.30) to compute $\theta_{\nu\lambda}p^{\nu\lambda}$, which is determined by the energy density flows:

$$\sum_K \frac{1}{\sqrt{gg_{oo}\gamma}}\frac{\partial\left(\sqrt{gg_{oo}\gamma}\mu\right)}{\partial s_K} = -\bar{p}_{\mu\lambda}\bar{\theta}^{\mu\lambda} = \bar{V}^{\mu}\bar{p}_{\mu}{}^{\lambda}{}_{;\lambda}. \tag{1.62}$$

We use the longitudinal conservation law Eq. (1.44):

$$V^{\mu}p_{\mu\nu;\lambda}g^{\nu\lambda}=\begin{pmatrix}\bar{g}^{\upsilon\upsilon'}\bar{V}^{j}\partial_{\upsilon}\bar{p}_{j\upsilon'}+\bar{g}^{\upsilon\upsilon'}\bar{V}^{o}\partial_{\upsilon}\bar{p}_{o\upsilon'}+\bar{g}^{\upsilon'\upsilon''}\bar{V}^{\upsilon}\partial_{\upsilon'}\bar{p}_{\upsilon\upsilon''}\\ -\bar{V}^{o}\bar{F}^{k}{}_{o\upsilon}\bar{p}_{k}{}^{\upsilon}-\bar{V}^{\upsilon}\bar{F}^{k}{}_{\upsilon o}\bar{p}_{k}{}^{o}-\bar{F}^{k}{}_{\upsilon\upsilon'}\bar{V}^{\upsilon}\bar{p}_{k}{}^{\upsilon'}\\ -\bar{p}_{o}{}^{\upsilon'}f_{\upsilon\upsilon'}\bar{V}^{\upsilon}\\ +\tfrac{1}{2}\bar{p}_{\upsilon''\upsilon'''}\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{\upsilon''\upsilon'''}+\tfrac{1}{2}\bar{p}_{oo}\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{oo}+\tfrac{1}{2}p_{jk}\bar{V}^{\upsilon}\partial_{\upsilon}\gamma^{jk}\end{pmatrix}. \quad (1.63)$$

We separate out the strategy and time components:

$$\sum_{K}\begin{pmatrix}-\frac{1}{\sqrt{\bar{g}\bar{g}_{oo}\gamma}}\frac{\partial\left(\sqrt{\bar{g}\bar{g}_{oo}\gamma}\mu\right)}{\partial s_K}\\ +\bar{g}^{\upsilon\upsilon'[K]}\bar{V}^{k[K]}\partial_{\upsilon}\bar{p}_{k[K]\upsilon'[K]}+\bar{g}^{\upsilon\upsilon'[K]}\bar{V}^{o}\partial_{\upsilon}\bar{p}_{o\upsilon'[K]}\\ +\bar{g}^{\upsilon'\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'}\bar{p}_{\upsilon[K]\upsilon''[K]}+\tfrac{1}{2}\bar{p}_{oo}\left[K\right]\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{oo}\\ -\bar{V}^{o}\bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{p}_{k[K]}{}^{\upsilon[K]}-\bar{V}^{\upsilon}\bar{F}^{k[K]}{}_{\upsilon o}\bar{p}_{k[K]}{}^{o}\\ -\bar{F}^{k[K]}{}_{\upsilon\upsilon'[K]}\bar{V}^{\upsilon}\bar{p}_{k[K]}{}^{\upsilon'[K]}-\bar{p}_{o}{}^{\upsilon'[K]}f_{\upsilon\upsilon'[K]}\bar{V}^{\upsilon}\\ +\tfrac{1}{2}\bar{p}_{\upsilon''[K]\upsilon'''[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{\upsilon''[K]\upsilon'''[K]}+\tfrac{1}{2}p_{j[K]k[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\gamma^{j[K]k[K]}\end{pmatrix}=0\,. \quad (1.64)$$

We express this directly in terms of the flows:

$$\sum_{K}\begin{pmatrix}-\frac{1}{\sqrt{\bar{g}\bar{g}_{oo}\gamma}}\frac{\partial\left(\sqrt{\bar{g}\bar{g}_{oo}\gamma}\mu\right)}{\partial s_K}+\frac{\partial\varphi_K}{\partial s_K}\\ -\bar{V}^{k[K]}\frac{\partial\left(\varphi_K\bar{V}_{k[K]}\right)}{\partial s_K}-\bar{V}^{o}\frac{\partial\left(\bar{\phi}\bar{V}_{o}\varphi_K\left(1-\bar{\psi}_K\right)\right)}{\partial s_K}\\ +\varphi_K\bar{g}^{\upsilon'\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'}\bar{g}_{\upsilon[K]\upsilon''[K]}-\bar{V}^{\upsilon[K]}\frac{\partial\left(\varphi_K\bar{V}_{\upsilon[K]}\right)}{\partial s_K}\\ +\varphi_K\bar{V}^{o}\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{V}^{\upsilon[K]}-\varphi_K\bar{\phi}\bar{V}^{o}\left(1-\bar{\psi}_K\right)\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon}\bar{V}^{\upsilon}\\ +\varphi_K\bar{V}^{\upsilon}\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon\upsilon'[K]}\bar{V}^{\upsilon'[K]}+\bar{\phi}\bar{V}_{o}\varphi_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon[K]}f_{\upsilon[K]\upsilon'[K]}\bar{V}^{\upsilon'[K]}\\ -\varphi_K\bar{V}^{\upsilon}\partial_{\upsilon}\ln\sqrt{\bar{g}\left[K\right]\gamma\left[K\right]}\\ -\bar{\phi}\varphi_K\bar{\psi}_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon}\partial_{\upsilon}\ln\sqrt{\bar{g}_{oo}}\\ -\tfrac{1}{2}\varphi_K\bar{V}_{\upsilon'[K]}\bar{V}_{\upsilon''[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{\upsilon'[K]\upsilon''[K]}-\tfrac{1}{2}\varphi_K\bar{V}_{j[K]}\bar{V}_{k[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\gamma^{j[K]k[K]}\end{pmatrix}=0\,. \quad (1.65)$$

We obtain the derivative of the time flow from the other flows:

$$
\begin{aligned}
&\tfrac{1}{2}\frac{\partial}{\partial s_K}\left(\bar{g}_{oo}\bar{V}^o\bar{V}^o\right)=\tfrac{1}{2}\frac{\partial \bar{g}_{oo}}{\partial s_K}\bar{V}^o\bar{V}^o+\frac{\partial \bar{V}^o}{\partial s_K}\bar{V}_o\\
&\frac{\partial \bar{V}^o}{\partial s_K}\bar{V}_o=\tfrac{1}{2}\frac{\partial}{\partial s_K}\left(\bar{g}_{oo}\bar{V}^o\bar{V}^o\right)-\tfrac{1}{2}\frac{\partial \bar{g}_{oo}}{\partial s_K}\bar{V}^o\bar{V}^o\\
&\frac{\partial \bar{V}^o}{\partial s_K}\bar{V}_o=\tfrac{1}{2}\frac{\partial}{\partial s_K}\left(1-\bar{g}_{\upsilon\upsilon'}\bar{V}^{\upsilon}\bar{V}^{\upsilon'}-\gamma_{jk}\bar{V}^j\bar{V}^k\right)-\tfrac{1}{2}\frac{\partial \bar{g}_{oo}}{\partial s_K}\bar{V}^o\bar{V}^o\ .\\
&\frac{\partial \bar{V}^o}{\partial s_K}\bar{V}_o=-\tfrac{1}{2}\frac{\partial \bar{g}_{oo}}{\partial s_K}\bar{V}^o\bar{V}^o-\tfrac{1}{2}\frac{\partial}{\partial s_K}\left(\bar{g}_{\upsilon\upsilon'}\bar{V}^{\upsilon}\bar{V}^{\upsilon'}+\gamma_{jk}\bar{V}^j\bar{V}^k\right)\\
&\frac{\partial \bar{V}^o}{\partial s_K}\bar{V}_o=\begin{pmatrix}-\tfrac{1}{2}\dfrac{\partial \bar{g}_{oo}}{\partial s_K}\bar{V}^o\bar{V}^o-\tfrac{1}{2}\dfrac{\partial \bar{g}_{\upsilon\upsilon'}}{\partial s_K}\bar{V}^{\upsilon}\bar{V}^{\upsilon'}\\ -\bar{V}_{\upsilon}\dfrac{\partial \bar{V}^{\upsilon}}{\partial s_K}-\tfrac{1}{2}\dfrac{\partial \gamma_{jk}}{\partial s_K}\bar{V}^j\bar{V}^k-\bar{V}_k\dfrac{\partial \bar{V}^k}{\partial s_K}\end{pmatrix}
\end{aligned}
\tag{1.66}
$$

Using this we obtain the following result:

$$
\sum_K\begin{pmatrix}
-\dfrac{1}{\sqrt{\bar{g}\bar{g}_{oo}\gamma}}\dfrac{\partial\left(\sqrt{\bar{g}\bar{g}_{oo}\gamma}\mu\right)}{\partial s_K}\\
+\varphi_K\bar{V}_{k[K]}\dfrac{\partial \bar{V}^{k[K]}}{\partial s_K}+\varphi_K\bar{V}_{\upsilon[K]}\dfrac{\partial \bar{V}^{\upsilon[K]}}{\partial s_K}-\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}_{\upsilon}\dfrac{\partial \bar{V}^{\upsilon}}{\partial s_K}+\bar{V}_k\dfrac{\partial \bar{V}^k}{\partial s_K}\right)\\
+\varphi_K\bar{V}^o\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{V}^{\upsilon[K]}-\varphi_K\bar{\phi}\bar{V}^o\left(1-\bar{\psi}_K\right)\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon}\bar{V}^{\upsilon}\\
+\varphi_K\bar{V}^{\upsilon}\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon\upsilon'[K]}\bar{V}^{\upsilon'[K]}+\bar{\phi}\bar{V}_o\varphi_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon}f_{\upsilon\upsilon'[K]}\bar{V}^{\upsilon'[K]}\\
+\varphi_K\bar{g}^{\upsilon'[K]\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'[K]}\bar{g}_{\upsilon[K]\upsilon''[K]}-\varphi_K\bar{V}^{\upsilon}\partial_{\upsilon}\ln\sqrt{\bar{g}[K]\gamma[K]}\\
-\bar{\phi}\varphi_K\bar{\psi}_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon}\partial_{\upsilon}\ln\sqrt{\bar{g}_{oo}}-\varphi_K\left(1-\bar{\psi}_K\right)\dfrac{\partial\ln\sqrt{\bar{g}_{oo}}}{\partial s_K}\\
-\tfrac{1}{2}\varphi_K\bar{V}_{\upsilon''[K]}\bar{V}_{\upsilon'''[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\bar{g}^{\upsilon''[K]\upsilon'''[K]}-\tfrac{1}{2}\varphi_K\bar{V}_{j[K]}\bar{V}_{k[K]}\bar{V}^{\upsilon}\partial_{\upsilon}\gamma^{j[K]k[K]}\\
+\tfrac{1}{2}\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\dfrac{\partial \bar{g}^{\upsilon\upsilon'}}{\partial s_K}\bar{V}_{\upsilon}\bar{V}_{\upsilon'}+\dfrac{\partial\gamma^{jk}}{\partial s_K}\bar{V}_j\bar{V}_k\right)
\end{pmatrix}=0\ . \tag{1.67}
$$

1.8.6 *Improved Results*

We can do a little better for the longitudinal and active flow equations by revisiting the divergence condition on the player flows in the covariant gauge:

$$\begin{aligned}
&\mathcal{S}_J{}^{\lambda}{}_{;\lambda} = \frac{1}{\sqrt{\bar{g}g_{oo}\gamma}}\partial_\upsilon\left(\sqrt{\bar{g}g_{oo}\gamma}\,\mathcal{S}_J{}^{\upsilon}\right)\\
&\mathcal{S}_J{}^{\lambda}{}_{;\lambda} = \frac{1}{\sqrt{\bar{g}g_{oo}\gamma}}\partial_{\upsilon[J]}\left(\sqrt{\bar{g}g_{oo}\gamma}\,\bar{V}^{\upsilon[J]}\right).\\
&\Rightarrow \mathcal{S}_J{}^{\lambda}{}_{;\lambda} = \bar{g}^{\upsilon[J]\upsilon'}\partial_{\upsilon[J]}\bar{V}_{\upsilon'} = 0
\end{aligned} \tag{1.68}$$

We see that our condition for the player flow is covariant. Moreover, we can use the gauge to evaluate one of the terms above:

$$\begin{aligned}
&\varphi_K\bar{g}^{\upsilon'[K]\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'[K]}\bar{g}_{\upsilon[K]\upsilon''[K]} = -\varphi_K\bar{V}_{\upsilon''[K]}\partial_{\upsilon'[K]}\bar{g}^{\upsilon'[K]\upsilon''[K]}\\
&\partial_{\upsilon'[K]}\bar{g}^{\upsilon'[K]\upsilon''[K]} = -\bar{g}^{\upsilon'[K]\upsilon''[K]}\partial_{\upsilon''[K]}\ln\sqrt{\bar{g}g_{oo}\gamma}\\
&\partial_{\upsilon'[K]}\bar{g}_{\upsilon[K]\upsilon''[K]}\bar{g}^{\upsilon'[K]\upsilon''[K]} = \partial_{\upsilon[K]}\ln\sqrt{\bar{g}g_{oo}\gamma}\\
&\varphi_K\bar{g}^{\upsilon'[K]\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'[K]}\bar{g}_{\upsilon[K]\upsilon''[K]} = \varphi_K\frac{\partial\ln\sqrt{\bar{g}g_{oo}\gamma}}{\partial s_K}
\end{aligned}\ . \tag{1.69}$$

We use this result as follows:

$$\begin{aligned}
&\begin{pmatrix}
+\varphi_K\bar{g}^{\upsilon'[K]\upsilon''[K]}\bar{V}^{\upsilon[K]}\partial_{\upsilon'[K]}\bar{g}_{\upsilon[K]\upsilon''[K]} - \varphi_K\bar{V}^{\upsilon}\partial_\upsilon\ln\sqrt{\bar{g}[K]\gamma[K]}\\
-\bar{\phi}\varphi_K\bar{\psi}_K(1-\bar{\psi}_K)\bar{V}^{\upsilon}\partial_\upsilon\ln\sqrt{\bar{g}_{oo}} - \varphi_K(1-\bar{\psi}_K)\frac{\partial\ln\sqrt{\bar{g}_{oo}}}{\partial s_K}
\end{pmatrix}\\
&= \begin{pmatrix}
+\varphi_K\bar{\psi}_K\frac{\partial\ln\sqrt{\bar{g}_{oo}}}{\partial s_K} - \bar{\phi}\varphi_K\bar{\psi}_K(1-\bar{\psi}_K)\bar{V}^{\upsilon}\partial_\upsilon\ln\sqrt{\bar{g}_{oo}}\\
+\varphi_K\frac{\partial\ln\sqrt{\bar{g}\gamma}}{\partial s_K} - \varphi_K\bar{V}^{\upsilon}\partial_\upsilon\ln\sqrt{\bar{g}[K]\gamma[K]}
\end{pmatrix}
\end{aligned}\ . \tag{1.70}$$

1.8.7 *Improved Active Flow Equations*

Using these results, we update the active flow Eq. (1.61):

$$\begin{pmatrix} +\varphi_J \bar{g}^{\upsilon'[J]\upsilon''[J]}\partial_{\upsilon'[J]}\bar{g}_{\upsilon[J]\upsilon''[J]} - \tfrac{1}{2}\left(\mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \bar{p}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} \\ -\tfrac{1}{2}\left(\mu\bar{V}_o\bar{V}_o - \bar{p}_{oo}\right)\partial_{\upsilon[J]}\bar{g}^{oo} - \tfrac{1}{2}\left(\mu V_k V_j - p_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk} \end{pmatrix} = \begin{pmatrix} +\varphi_J\partial_{\upsilon[J]}\ln\sqrt{\bar{g}\bar{g}_{oo}\gamma} - \tfrac{1}{2}\left(\mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \bar{p}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} \\ -\tfrac{1}{2}\left(\mu\bar{V}_o\bar{V}_o - \bar{p}_{oo}\right)\partial_{\upsilon[J]}\bar{g}^{oo} - \tfrac{1}{2}\left(\mu V_k V_j - p_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk} \end{pmatrix}. \tag{1.71}$$

We substitute the indicated stress contributions:

$$\begin{aligned} &\begin{pmatrix} +\varphi_J \bar{g}^{\upsilon'[J]\upsilon''[J]}\partial_{\upsilon'[J]}\bar{g}_{\upsilon[J]\upsilon''[J]} - \tfrac{1}{2}\left(\mu\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \bar{p}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} \\ -\tfrac{1}{2}\left(\mu\bar{V}_o\bar{V}_o - \bar{p}_{oo}\right)\partial_{\upsilon[J]}\bar{g}^{oo} - \tfrac{1}{2}\left(\mu V_k V_j - p_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk} \end{pmatrix} \\ &= \begin{pmatrix} -\tfrac{1}{2}\left(\left(\mu+\varphi_J\right)\bar{V}_{\upsilon'}\bar{V}_{\upsilon''} - \varphi_J\bar{g}_{\upsilon''\upsilon'}\right)\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} - \tfrac{1}{2}\left(\left(\mu+\varphi_J\right)V_k V_j - \varphi_J\gamma_{jk}\right)\partial_{\upsilon[J]}\gamma^{jk} \\ +\varphi_J\partial_{\upsilon[J]}\ln\sqrt{\bar{g}\gamma} + \left(\mu\bar{V}_o\bar{V}^o + \varphi_J\left(1-\bar{\phi}\bar{\psi}_J\left(1-\bar{\psi}_J\right)\right)\right)\partial_{\upsilon[J]}\ln\sqrt{\bar{g}_{oo}} \end{pmatrix} \\ &= \begin{pmatrix} -\tfrac{1}{2}\left(\mu+\varphi_J\right)\bar{V}_{\upsilon'}\bar{V}_{\upsilon''}\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} - \tfrac{1}{2}\left(\mu+\varphi_J\right)V_k V_j\partial_{\upsilon[J]}\gamma^{jk} \\ +\left(\mu\bar{V}_o\bar{V}^o + \varphi_J\left(1-\bar{\phi}\bar{\psi}_J\left(1-\bar{\psi}_J\right)\right)\right)\partial_{\upsilon[J]}\ln\sqrt{\bar{g}_{oo}} \end{pmatrix} \end{aligned}. \tag{1.72}$$

This updates the active contribution:

$$\sum_K \mu\frac{\partial V_{\upsilon[J]}}{\partial s_K} + \varphi_J\frac{\partial\bar{V}_{\upsilon[J]}}{\partial s_J} = \begin{pmatrix} \partial_{\upsilon[J]}\varphi_J - \bar{V}_{\upsilon[J]}\bar{V}^{\upsilon'[J]}\partial_{\upsilon'[J]}\varphi_J + \bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\ -\tfrac{1}{2}\left(\mu+\varphi_J\right)\bar{V}_{\upsilon'}\bar{V}_{\upsilon''}\partial_{\upsilon[J]}\bar{g}^{\upsilon''\upsilon'} - \tfrac{1}{2}\left(\mu+\varphi_J\right)V_k V_j\partial_{\upsilon[J]}\gamma^{jk} \\ +\left(\mu\bar{V}_o\bar{V}^o + \varphi_J\left(1-\bar{\phi}\bar{\psi}_J\left(1-\bar{\psi}_J\right)\right)\right)\partial_{\upsilon[J]}\ln\sqrt{\bar{g}_{oo}} \\ +\sum_K \bar{V}_o\bar{\phi}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)\bar{f}_{\upsilon[J]\upsilon'[K]}\bar{V}^{\upsilon'[K]} \\ +\sum_K\left(\varphi_K\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'[K]} + \sum_L \mu V_{l[L]}\bar{F}^{l[L]}{}_{\upsilon[J]\upsilon'[K]}\right)\bar{V}^{\upsilon'[K]} \\ -\sum_K \bar{\phi}\bar{V}^o\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]} \end{pmatrix} \tag{1.73}$$

1.8.8 *Improved Longitudinal Equations*

The flow gradients in Eq. (1.67) are:

$$
\begin{aligned}
&\left(+\varphi_K \bar{V}_{k[K]}\frac{\partial \bar{V}^{k[K]}}{\partial s_K} + \varphi_K \bar{V}_{\upsilon[K]}\frac{\partial \bar{V}^{\upsilon[K]}}{\partial s_K} - \bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}_\upsilon \frac{\partial \bar{V}^\upsilon}{\partial s_K} + \bar{V}_k \frac{\partial \bar{V}^k}{\partial s_K}\right)\right) \\
&= \begin{pmatrix} +\varphi_K \bar{V}^{k[K]}\frac{\partial \bar{V}_{k[K]}}{\partial s_K} + \varphi_K \bar{V}^{\upsilon[K]}\frac{\partial \bar{V}_{\upsilon[K]}}{\partial s_K} + \varphi_K \bar{V}_{k[K]}\bar{V}_{j[K]}\frac{\partial \bar{g}^{k[K]j[K]}}{\partial s_K} + \varphi_K \bar{V}_{\upsilon[K]}\bar{V}_{\upsilon'[K]}\frac{\partial \bar{g}^{\upsilon[K]\upsilon'[K]}}{\partial s_K} \\ -\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}^\upsilon \frac{\partial \bar{V}_\upsilon}{\partial s_K} + \bar{V}^k \frac{\partial \bar{V}_k}{\partial s_K}\right) - \bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}_\upsilon \bar{V}_{\upsilon'}\frac{\partial \bar{g}^{\upsilon\upsilon'}}{\partial s_K} + \bar{V}_j\bar{V}_k\frac{\partial \gamma^{jk}}{\partial s_K}\right) \end{pmatrix}.
\end{aligned}
\tag{1.74}
$$

The result for the longitudinal equation is:

$$
\frac{1}{\sqrt{\bar{g}\bar{g}_{oo}}\gamma}\sum_K \frac{\partial\left(\sqrt{\bar{g}\bar{g}_{oo}}\gamma\mu\right)}{\partial s_K} = \sum_K \begin{pmatrix}
\varphi_K \bar{V}^o \bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{V}^{\upsilon[K]} - \varphi_K \bar{\phi}\bar{V}^o\left(1-\bar{\psi}_K\right)\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon}\bar{V}^\upsilon \\
-\varphi_K \bar{V}^{\upsilon'[K]}\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon'[K]\upsilon}\bar{V}^\upsilon - \bar{\phi}\bar{V}_o\varphi_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon'[K]}f_{\upsilon'[K]\upsilon}\bar{V}^\upsilon \\
+\varphi_K \bar{V}^{k[K]}\frac{\partial \bar{V}_{k[K]}}{\partial s_K} + \varphi_K \bar{V}^{\upsilon[K]}\frac{\partial \bar{V}_{\upsilon[K]}}{\partial s_K} - \bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}^\upsilon\frac{\partial \bar{V}_\upsilon}{\partial s_K} + \bar{V}^k\frac{\partial \bar{V}_k}{\partial s_K}\right) \\
+\varphi_K\bar{\psi}_K\frac{\partial \ln\sqrt{\bar{g}_{oo}}}{\partial s_K} - \bar{\phi}\varphi_K\bar{\psi}_K\left(1-\bar{\psi}_K\right)\bar{V}^\upsilon\partial_\upsilon \ln\sqrt{\bar{g}_{oo}} \\
+\varphi_K\frac{\partial \ln\sqrt{\bar{g}}\gamma}{\partial s_K} - \varphi_K\bar{V}^\upsilon\partial_\upsilon \ln\sqrt{\bar{g}[K]\gamma[K]} \\
-\tfrac{1}{2}\varphi_K\bar{V}_{\upsilon''[K]}\bar{V}_{\upsilon'''[K]}\bar{V}^\upsilon\partial_\upsilon\bar{g}^{\upsilon''[K]\upsilon'''[K]} - \tfrac{1}{2}\varphi_K\bar{V}_{j[K]}\bar{V}_{k[K]}\bar{V}^\upsilon\partial_\upsilon\gamma^{j[K]k[K]} \\
+\varphi_K\bar{V}_{k[K]}\bar{V}_{j[K]}\frac{\partial \bar{g}^{k[K]j[K]}}{\partial s_K} + \varphi_K\bar{V}_{\upsilon[K]}\bar{V}_{\upsilon'[K]}\frac{\partial \bar{g}^{\upsilon[K]\upsilon'[K]}}{\partial s_K} \\
-\tfrac{1}{2}\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\frac{\partial \bar{g}^{\upsilon\upsilon'}}{\partial s_K}\bar{V}_\upsilon\bar{V}_{\upsilon'} + \frac{\partial \gamma^{jk}}{\partial s_K}\bar{V}_j\bar{V}_k\right)
\end{pmatrix}. \tag{1.75}
$$

This separates the derivatives of the independent variables from the derivatives of the metric.

1.9 Pressure Contributions and Constant Metric

To use the results for the longitudinal equation for the energy density and the equations for the transverse flows, we assume that the player pressures φ_K, the metric elements for the active strategies $\bar{g}_{\upsilon\upsilon'}$, the inactive strategies γ_{jk}, the time component $\bar{g}_{oo}$ and payoff strategies $\left\{\bar{F}^k{}_{\upsilon\upsilon'}\ \bar{F}^k{}_{\upsilon o}\ f_{\upsilon\upsilon'}\right\}$ are given functions of the player preferences[1].

We start by exploring the contributions of the pressure, which on a per player basis reflects constraints that each player may experience. For example, each player may experience a resistance as their effort

[1] In our initial computations we have taken these to be constants.

increases, especially with respect to another player. Such resistance can be captured in the player's pressure. Another possibility is that the player effort becomes so small or another player's effort becomes so large that again there will be some resistance. These are constraints that are external to the strategies that are considered active in the sense that they may arise from player's that are currently being ignored.

For current numerical work, we can assume that all of the metric elements are constant. Based on these assumptions, the energy density, Eq. (1.75), is:

$$\sum_K \frac{\partial \mu}{\partial s_K} = \sum_K \begin{pmatrix} -\bar{\phi}\varphi_K \sum_{L\neq K}\left(1-\bar{\psi}_K\right)\left(\bar{V}^{k[L]}\frac{\partial \bar{V}_{k[L]}}{\partial s_K} + \bar{V}^{\upsilon[L]}\frac{\partial \bar{V}_{\upsilon[L]}}{\partial s_K}\right) \\ -\bar{\phi}\varphi_K \sum_{L\neq K}\bar{\psi}_L\left(\bar{V}^{k[K]}\frac{\partial \bar{V}_{k[K]}}{\partial s_K} + \bar{V}^{\upsilon[K]}\frac{\partial \bar{V}_{\upsilon[K]}}{\partial s_K}\right) \\ -\bar{\phi}\varphi_K \bar{V}^{o}\bar{V}_{k[K]}\sum_{L\neq K}\left(\bar{\psi}_L \bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{V}^{\upsilon[K]} + \left(1-\bar{\psi}_K\right)\bar{F}^{k[K]}{}_{o\upsilon[L]}\bar{V}^{\upsilon[L]}\right) \\ -\varphi_K \bar{V}_{k[K]}\bar{V}^{\upsilon'[K]}\bar{F}^{k[K]}{}_{\upsilon'[K]\upsilon}\bar{V}^{\upsilon} - \bar{\phi}\bar{V}_o\varphi_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon'[K]}f_{\upsilon'[K]\upsilon}\bar{V}^{\upsilon} \end{pmatrix}. \quad (1.76)$$

The inactive flow, Eq. (1.57), is:

$$\left(\mu+\varphi_J\right)\frac{\partial V_{j[J]}}{\partial s_J} + \sum_{K\neq J}\mu\frac{\partial V_{j[J]}}{\partial s_K} = -V_{j[J]}\frac{\partial \varphi_J}{\partial s_J} + V_{j[J]}\theta_{\nu\lambda}p^{\nu\lambda}. \quad (1.77)$$

The active flow, Eq. (1.73), is:

$$\sum_K \mu\frac{\partial V_{\upsilon[J]}}{\partial s_K} + \varphi_J\frac{\partial \bar{V}_{\upsilon[J]}}{\partial s_J} = \begin{pmatrix} \partial_{\upsilon[J]}\varphi_J - \bar{V}_{\upsilon[J]}\frac{\partial \varphi_J}{\partial s_J} + \bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda} \\ +\sum_K \bar{V}_o\bar{\phi}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)\bar{f}_{\upsilon[J]\upsilon'[K]}\bar{V}^{\upsilon'[K]} \\ +\sum_K\left(\varphi_K\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'[K]} + \sum_L \mu V_{l[L]}\bar{F}^{l[L]}{}_{\upsilon[J]\upsilon'[K]}\right)\bar{V}^{\upsilon'[K]} \\ -\sum_K \bar{\phi}\bar{V}^{o}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right) - \sum_{L\neq K}\mu\bar{\psi}_L\right)V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]} \end{pmatrix}. \quad (1.78)$$

The time flow, Eq. (1.58), is:

$$\begin{pmatrix} \sum_K \left(\mu + \varphi_K \bar{\phi}\,(1-\bar{\psi}_K)\right) \dfrac{\partial \bar{V}_o}{\partial s_K} \\ +\bar{V}_o \sum_K \dfrac{\partial\left(\varphi_K \bar{\phi}\,(1-\bar{\psi}_K)\right)}{\partial s_K} - \bar{V}_o \bar{\theta}_{\nu\lambda} \bar{p}^{\nu\lambda} \\ -\sum_{K,L} \bar{F}^{k[K]}{}_{o\upsilon[L]} \left(\mu V_{k[K]} \bar{V}^{\upsilon[L]} + \delta_{KL} \varphi_K \bar{V}_{k[K]} \bar{V}^{\upsilon[K]}\right) \end{pmatrix} = 0 . \tag{1.79}$$

In practice, we may use the fact that the flow vector is a unit vector and conserved along the streamlines as a replacement for the time flow equation.

1.10 Frame Contributions with Reduced Constant Metric

In addition to the pressure contribution, we may also have frame dependent forces. To give some sense about these, it is useful to recall that behaviors may appear simpler in a frame that is in some sense, co-moving. We expect that there will be frames in which the forces will be unassociated with acceleration effects.

In looking for full solutions to the equations, these concepts are made precise. For numerical work, we suggest a simplification in which the time component of the metric and space components of the metric are:

$$\begin{aligned} \bar{g}_{oo} &= e^{2\nu} \\ \bar{g}_{\upsilon\upsilon'} &= -e^{2\Phi} \delta_{\upsilon\upsilon'} \end{aligned} . \tag{1.80}$$

To use the equations we have developed, we impose the covariant gauge, which imposes the following conditions:

$$\begin{aligned} &\partial_{\upsilon'}\left(\bar{g}^{\upsilon'\upsilon} \sqrt{\bar{g}\gamma\bar{g}_{oo}}\right) = 0 \\ &n = n_a + n_i \\ &-2\Phi + n\Phi + \nu = 0 \Rightarrow \Phi = -\frac{\nu}{n-2} \\ &\partial_o \nu = 0 \end{aligned} . \tag{1.81}$$

Using these results, we obtain the various flow equations.

The inactive flow equations are formally unchanged:

$$\left(\mu+\varphi_J\right)\frac{\partial V_{j[J]}}{\partial s_J}+\sum_{K\neq J}\mu\frac{\partial V_{j[J]}}{\partial s_K}=-V_{j[J]}\frac{\partial\varphi_J}{\partial s_J}+V_{j[J]}\theta_{\nu\lambda}p^{\nu\lambda}. \qquad (1.82)$$

The active flow equations will be changed. We note first that the contributions to the time component can be written in terms of the same reduced quantities that occur in Minkowski space:

$$\begin{aligned}\bar{\psi}_J &= \bar{V}_{\upsilon[J]}\bar{V}^{\upsilon[J]}+V_{j[J]}V^{j[J]}\\ \bar{\psi}_J &= -e^{-2\Phi}\left(\bar{V}_{\upsilon'[J]}\bar{V}_{\upsilon[J]}\delta^{\upsilon'[J]\upsilon[J]}+V_{j[J]}V_{k[J]}\delta^{j[J]k[J]}\right).\\ \bar{\psi}_J &\equiv e^{-2\Phi}\hat{\psi}_J\end{aligned} \qquad (1.83)$$

Using this we obtain the following for the active flow equations:

$$\sum_K\mu\frac{\partial V_{\upsilon[J]}}{\partial s_K}+\varphi_J\frac{\partial\bar{V}_{\upsilon[J]}}{\partial s_J}=\begin{pmatrix}\partial_{\upsilon[J]}\varphi_J-\bar{V}_{\upsilon[J]}\bar{V}^{\upsilon'[J]}\partial_{\upsilon'[J]}\varphi_J+\bar{V}_{\upsilon[J]}\bar{\theta}_{\nu\lambda}\bar{p}^{\nu\lambda}\\ +\left(\mu+\varphi_J\right)\left(1-\bar{V}^o\bar{V}_o\right)\partial_{\upsilon[J]}\Phi\\ +\left(\mu\bar{V}_o\bar{V}^o+\varphi_J\left(1-\bar{\phi}\bar{\psi}_J\left(1-\bar{\psi}_J\right)\right)\right)\partial_{\upsilon[J]}\nu\\ +\sum_K\bar{V}_o\bar{\phi}\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right)-\sum_{L\neq K}\mu\bar{\psi}_L\right)\bar{f}_{\upsilon[J]\upsilon'[K]}\bar{V}^{\upsilon'[K]}\\ +\sum_K\left(\left(\mu+\varphi_K\right)\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon[J]\upsilon'[K]}+\sum_{L\neq K}\mu V_{l[L]}\bar{F}^{l[L]}{}_{\upsilon[J]\upsilon'[K]}\right)\bar{V}^{\upsilon'[K]}\\ -\sum_K\bar{\phi}\bar{V}^o\left(\left(\mu+\varphi_K\right)\left(1-\bar{\psi}_K\right)-\sum_{L\neq K}\mu\bar{\psi}_L\right)V_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[J]}\end{pmatrix}. \qquad (1.84)$$

The last contribution is the longitudinal conservation equation:

$$\sum_K\left(\frac{\partial\mu}{\partial s_K}+2\mu\frac{\partial\Phi}{\partial s_K}\right)=\sum_K\begin{pmatrix}\varphi_K\bar{V}^o\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon[K]}\bar{V}^{\upsilon[K]}-\varphi_K\bar{\phi}\bar{V}^o\left(1-\bar{\psi}_K\right)\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{o\upsilon}\bar{V}^{\upsilon}\\ -\varphi_K\bar{V}^{\upsilon'[K]}\bar{V}_{k[K]}\bar{F}^{k[K]}{}_{\upsilon'[K]\upsilon}\bar{V}^{\upsilon}-\bar{\phi}\bar{V}_o\varphi_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon'[K]}f_{\upsilon'[K]\upsilon}\bar{V}^{\upsilon}\\ +\varphi_K\bar{V}^{k[K]}\frac{\partial\bar{V}_{k[K]}}{\partial s_K}+\varphi_K\bar{V}^{\upsilon[K]}\frac{\partial\bar{V}_{\upsilon[K]}}{\partial s_K}-\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(\bar{V}^{\upsilon}\frac{\partial\bar{V}_{\upsilon}}{\partial s_K}+\bar{V}^k\frac{\partial\bar{V}_k}{\partial s_K}\right)\\ +\varphi_K\bar{\psi}_K\frac{\partial\nu}{\partial s_K}-\bar{\phi}\varphi_K\bar{\psi}_K\left(1-\bar{\psi}_K\right)\bar{V}^{\upsilon}\partial_{\upsilon}\nu-\varphi_K\left(n[K]-\bar{\psi}_K\right)\bar{V}^{\upsilon}\partial_{\upsilon}\Phi\\ +\varphi_K\left(n-2\bar{\psi}_K\right)\frac{\partial\Phi}{\partial s_K}+\bar{\phi}\varphi_K\left(1-\bar{\psi}_K\right)\left(1-\bar{V}^o\bar{V}_o\right)\frac{\partial\Phi}{\partial s_K}\end{pmatrix}. \qquad (1.85)$$

We can also express this as:

$$\sum_K \left(\frac{\partial \mu}{\partial s_K} + 2\frac{\partial \Phi}{\partial s_K} \right) = \sum_K \left(\begin{array}{l} -\bar{\phi}\bar{V}^o \varphi_K \bar{V}_{k[K]} \sum_{L \neq K} \left(\bar{\psi}_L \bar{F}^{k[K]}{}_{o\upsilon[K]} \bar{V}^{\upsilon[K]} + \left(1-\bar{\psi}_K\right) \bar{F}^{k[K]}{}_{o\upsilon[L]} \bar{V}^{\upsilon[L]} \right) \\ -\varphi_K \bar{V}^{\upsilon'[K]} \bar{V}_{k[K]} \bar{F}^{k[K]}{}_{\upsilon'[K]\upsilon} \bar{V}^{\upsilon} - \bar{\phi}\bar{V}_o \varphi_K \left(1-\bar{\psi}_K\right) \bar{V}^{\upsilon'[K]} f_{\upsilon'[K]\upsilon} \bar{V}^{\upsilon} \\ +\varphi_K \bar{V}^{k[K]} \frac{\partial \bar{V}_{k[K]}}{\partial s_K} + \varphi_K \bar{V}^{\upsilon[K]} \frac{\partial \bar{V}_{\upsilon[K]}}{\partial s_K} - \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right) \left(\bar{V}^{\upsilon} \frac{\partial \bar{V}_{\upsilon}}{\partial s_K} + \bar{V}^{k} \frac{\partial \bar{V}_{k}}{\partial s_K} \right) \\ +\varphi_K \bar{\psi}_K \frac{\partial \nu}{\partial s_K} - \bar{\phi}\varphi_K \bar{\psi}_K \left(1-\bar{\psi}_K\right) \bar{V}^{\upsilon} \partial_{\upsilon} \nu - \varphi_K \left(n[K] - \bar{\psi}_K \right) \bar{V}^{\upsilon} \partial_{\upsilon} \Phi \\ +\varphi_K \left(n - 2\bar{\psi}_K \right) \frac{\partial \Phi}{\partial s_K} + \bar{\phi}\varphi_K \left(1-\bar{\psi}_K\right) \left(1 - \bar{V}^o \bar{V}_o\right) \frac{\partial \Phi}{\partial s_K} \end{array} \right) \tag{1.86}$$

The frame dependence, in analogy to gravitational forces, may be related to areas of high energy density. We need to plot the energy density to see whether this might be plausible.

1.11 Seasonal Payoffs

The next effect we include will be the ***seasonal payoffs***, $f_{\upsilon\upsilon'}$. We have computed them above, but we did not specify what values they might take. We now say a word about the possible values and the effects we have in mind.

One possibility is that the effects are like the impact on weather phenomena due to the changes in season. Seasonal changes are due to the periodic motion of the earth around the sun. There are of course other periodic changes, such as the daily changes due to the rotation of the earth.

Decision making is not immune to such effects. Our calendar of special events reflects such seasonal impacts. We also impose arbitrary "seasons" such quarterly earnings, which generate periodic changes in stock prices. Companies focus each quarter on making their earnings look as good as possible. These fluctuations we interpret as frame dependent processes.

What might reasonable functions be for the seasonal payoffs? One model is to imagine that the seasonal payoffs between any two players is the same constant, for example ω independent of the strategy each player may choose. We produce such a result with ***seasonal vector potentials***:

$$\begin{aligned} &a_{\upsilon[J]} = \alpha_J \left(\forall K : r_K\right) \\ &r_K = \sum_{\upsilon[K]} y^{\upsilon[K]} \end{aligned} \quad . \tag{1.87}$$

We then compute the associated *seasonal payoff*:

$$\begin{aligned} &f_{\upsilon[J]\upsilon'[K]} = \partial_{\upsilon[J]} a_{\upsilon'[K]} - \partial_{\upsilon'[K]} a_{\upsilon[J]} \\ &J = K \Rightarrow f_{\upsilon[J]\upsilon'[J]} = \partial_{\upsilon[J]} \alpha_J - \partial_{\upsilon'[J]} \alpha_J = 0 \\ &J \neq K \Rightarrow f_{\upsilon[J]\upsilon'[K]} = \partial_{\upsilon[J]} \alpha_K - \partial_{\upsilon'[K]} \alpha_J = \frac{\partial \alpha_K}{\partial r_J} - \frac{\partial \alpha_J}{\partial r_K} \end{aligned} \quad . \tag{1.88}$$

In addition we also have the covariant gauge that constrains the values we can pick:

$$\begin{aligned} &\frac{1}{\sqrt{gg_{oo}\gamma}} \partial_\upsilon \left(g^{\upsilon o} \sqrt{gg_{oo}\gamma} \right) = 0 \\ &\Rightarrow \frac{1}{\sqrt{gg_{oo}\gamma}} \partial_\upsilon \left(\overline{g}^{\upsilon\upsilon'} a_{\upsilon'} \sqrt{gg_{oo}\gamma} \right) = 0 \\ &\Rightarrow \frac{1}{\sqrt{gg_{oo}\gamma}} \partial_\upsilon \left(\overline{g}^{\upsilon\upsilon'} \sqrt{gg_{oo}\gamma} \right) a_{\upsilon'} + \partial_\upsilon a_{\upsilon'} = 0 \, . \\ &\Rightarrow g^{\upsilon\upsilon'} \partial_\upsilon a_{\upsilon'} = 0 \\ &\sum_J \left(\sum_{\upsilon[J],\upsilon'[J]} g^{\upsilon[J]\upsilon'[J]} \right) \alpha_J = 0 \end{aligned} \tag{1.89}$$

This constraint also does not depend on any specific set of player strategies, only composite properties.

In other words, a natural choice is that for each pair of players, the *seasonal payoff* is a constant. This choice of seasonal payoff favors no particular player strategy. That doesn't mean however that we believe this is the only choice. One is free to pick other choices. We pursue here however this simple model. Using Eq. (1.1) we compute the gauge contribution $G^k{}_{\upsilon\upsilon'}$:

$$\begin{aligned} \bar{F}^k_{\upsilon\upsilon'} &= F^k{}_{\upsilon\upsilon'} + a_\upsilon F^k{}_{\upsilon' o} - a_{\upsilon'} F^k{}_{\upsilon o} \equiv F^k{}_{\upsilon\upsilon'} + G^k{}_{\upsilon\upsilon'} \\ G^k{}_{\upsilon[J]\upsilon'[K]} &= a_{\upsilon[J]} F^k{}_{\upsilon'[K]o} - a_{\upsilon'[K]} F^k{}_{\upsilon[J]o} \\ G^k{}_{\upsilon[J]\upsilon'[K]} &= \alpha_J F^k{}_{\upsilon'[K]o} - \alpha_K F^k{}_{\upsilon[J]o} \end{aligned} \quad . \qquad (1.90)$$

For a choice of games that we call zero-sum, we have

$$\begin{aligned} F^k{}_{\upsilon'[K]o} &\equiv F^k{}_K \\ G^k{}_{\upsilon[J]\upsilon'[K]} &= \alpha_J F^k{}_K - \alpha_K F^k{}_J \ . \\ G^k{}_{\upsilon[J]\upsilon'[J]} &= 0 \end{aligned} \qquad (1.91)$$

In particular for two-person games and specific choices for the *seasonal vector potentials*, we have:

$$\begin{aligned} & J \neq K \\ & F^k{}_K = -F^k{}_J \\ & \alpha_1 = -\tfrac{1}{2}\omega r_2 \\ & \alpha_2 = \tfrac{1}{2}\omega r_1 \\ & G^k{}_{\upsilon[1]\upsilon'[2]} = (\alpha_1 + \alpha_2) F^k{}_2 = \tfrac{1}{2}\omega(r_1 - r_2) F^k{}_2 \end{aligned} \quad . \qquad (1.92)$$

This is an interesting result: for each fixed set of preferences, the *game matrix* in the *time orthogonal frame* differs from the original by a constant independent of the strategies. This says that the two games are in fact the same when looked at using classic game theory. In addition, the functional dependence is the difference between the efforts, consistent with having the total effort $r_1 + r_2$ be inactive.

An important observation is that such periodic *seasonal payoffs* may be a way to study the structure of decisions. By looking at systems in different periodic situations, we might be able to verify such structures.

1.12 Stationary Harmonic Flows

So far we have described one set of stationary harmonic flows. These result from the payoff structures of the players, any seasonal payoffs and a model for the *drift* frame contribution and the pressure constraints for an ideal player fluid. Stationary flows result after all transient effects have died out. These particular stationary flows are in a frame in which the metric is independent of time.

One additional type of stationary flow results from perturbing the system with an impulse at some initial point of time. These flows in general will be in a frame in which the metric is time dependent. In general, we show that an initial impulse or perturbation can be expressed as a linear combination over (an infinite) set of harmonic contributions. We study this full set of ***harmonic perturbations*** on the system.

The way we approach harmonic perturbations is to start with our initial active metric g^{ab} and inactive metric γ_{jk} structures in the *holonomic stationary frame*, in which these metric components are independent of time. Let the coordinates for this frame be y^a.

Realistic behaviors will have metric components that are time dependent. Assume that we have such a metric and choose the covariant gauge with coordinates x^a. It follows that these coordinates satisfy the wave equation. We write this wave equation in the *holonomic stationary frame* we started with:

$$\begin{aligned}
&g^{\mu\nu}x^a{}_{;\mu\nu}=0\\
&g^{cd}x^a{}_{;cd}+\gamma^{jk}x^a{}_{;jk}=0\\
&g^{cd}\partial_d\left(x^a{}_{;c}\right)-g^{cd}\partial_e x^a\varpi^e{}_{cd}+\partial_e x^a g^{ef}\partial_f\ln\sqrt{\gamma}=0\\
&g^{cd}x^a{}_{\backslash cd}+\partial_e x^a g^{ef}\partial_f\ln\sqrt{\gamma}=0
\end{aligned}. \tag{1.93}$$

We now express the result in the *time orthogonal frame*:

$$\begin{aligned}&\bar{g}^{\upsilon\upsilon'}x^a{}_{|\upsilon\upsilon'}+\bar{g}^{oo}x^a{}_{|oo}+\partial_e x^a g^{ef}\partial_f \ln\sqrt{\gamma}=0\\&\begin{pmatrix}\bar{g}^{\upsilon\upsilon'}\partial_{\upsilon'}\partial_\upsilon x^a+\partial_{\upsilon'}\bar{g}^{\upsilon\upsilon'}\partial_\upsilon x^a\\+\partial_\upsilon x^a\bar{g}^{\upsilon\upsilon'}\partial_{\upsilon'}\ln\sqrt{\bar{g}\bar{g}_{oo}\gamma}+\bar{g}^{oo}\partial_o\partial_o x^a\end{pmatrix}=0\\&\frac{1}{\sqrt{\bar{g}\bar{g}_{oo}\gamma}}\partial_{\upsilon'}\left(\bar{g}^{\upsilon\upsilon'}\sqrt{\bar{g}\bar{g}_{oo}\gamma}\partial_\upsilon x^a\right)+\bar{g}^{oo}\partial_o\partial_o x^a=0\end{aligned} \quad . \qquad (1.94)$$

Assuming the metric in the *time orthogonal frame* satisfies the covariant gauge, we get the wave equation:

$$\bar{g}^{\upsilon\upsilon'}\partial_{\upsilon'}\partial_\upsilon x^a+\bar{g}^{oo}\partial_o\partial_o x^a=0\,. \qquad (1.95)$$

We now assume a harmonic perturbation for the strategic preferences:

$$\begin{aligned}&x^{\bar{\upsilon}}=x^{\bar{\upsilon}}{}_\omega e^{i\omega t}\\&\omega^2 x^{\bar{\upsilon}}{}_\omega-\bar{g}_{oo}\bar{g}^{\upsilon\upsilon'}\partial_{\upsilon'}\partial_\upsilon x^{\bar{\upsilon}}{}_\omega=0\end{aligned}\,. \qquad (1.96)$$

We make the frame assumption Eq. (1.80):

$$\begin{aligned}&\delta^{\upsilon\upsilon'}\partial_{\upsilon'}\partial_\upsilon x^{\bar{\upsilon}}{}_\omega+\omega^2 e^{-2(\nu-\Phi)}x^{\bar{\upsilon}}{}_\omega=0\\&\delta^{\upsilon\upsilon'}\partial_{\upsilon'}\partial_\upsilon x^{\bar{\upsilon}}{}_\omega+\omega^2 e^{-2\frac{n-1}{n-2}\nu}x^{\bar{\upsilon}}{}_\omega=0\end{aligned}\,. \qquad (1.97)$$

This looks like a Laplace equation that we can solve numerically. The challenge may be that Mathematica may not have a general purpose solver using Dirichlet conditions for dimensions higher than three.

Note that we can make the frequency for each coordinate different, corresponding to different impulses along each strategic preference direction.

1.13 Spectral Decomposition

The wave equation can be decomposed in a spectral decomposition, which helps in special cases. We consider one such case in which the frame effects depend only on the player efforts r_J. We make a transformation to a coordinate frame to reflect this:

$$r_J = \frac{1}{n_J}\sum_{j[J]} y_{j[J]}$$
$$q_{j[J]} = y_{j[J]} - r_J \;. \tag{1.98}$$
$$\sum_{j[J]} q_{j[J]} = 0$$

We express the gradients of any scalar as follows:

$$\frac{\partial\varphi}{\partial y_{j[J]}} = \frac{1}{n_J}\frac{\partial\varphi}{\partial r_J} + \frac{\partial\varphi}{\partial q_{j[J]}}\left(1-\frac{1}{n_J}\right)$$
$$q_{n_J} = -\sum_{j[J]\neq n_J} q_{j[J]} \qquad . \tag{1.99}$$
$$\frac{\partial\varphi}{\partial y_{n_J}} = \frac{1}{n_J}\frac{\partial\varphi}{\partial r_J} - \left(1-\frac{1}{n_J}\right)\sum_{j[J]\neq n_J}\frac{\partial\varphi}{\partial q_{j[J]}}$$

We compute the second derivatives:

$$\frac{\partial^2\varphi}{\partial {y_{j[J]}}^2} = \frac{1}{{n_J}^2}\frac{\partial^2\varphi}{\partial {r_J}^2} + \frac{2}{n_J}\frac{\partial^2\varphi}{\partial r_J \partial q_{j[J]}}\left(1-\frac{1}{n_J}\right) + \frac{\partial^2\varphi}{\partial {q_{j[J]}}^2}\left(1-\frac{1}{n_J}\right)^2$$
$$\frac{\partial^2\varphi}{\partial {y_{n_J}}^2} = \frac{1}{{n_J}^2}\frac{\partial^2\varphi}{\partial {r_J}^2} - \frac{2}{n_J}\left(1-\frac{1}{n_J}\right)\sum_{j[J]\neq n_J}\frac{\partial^2\varphi}{\partial r_J \partial q_{j[J]}} + \left(1-\frac{1}{n_J}\right)^2 \sum_{j[J],k[J]\neq n_J}\frac{\partial^2\varphi}{\partial q_{k[J]}\partial q_{j[J]}} \;. \tag{1.100}$$
$$\sum\frac{\partial^2\varphi}{\partial {y_{j[J]}}^2} = \frac{1}{n_J}\frac{\partial^2\varphi}{\partial {r_J}^2} + \left(1-\frac{1}{n_J}\right)^2 \sum_{j[J]\neq n_J}\left(\frac{\partial^2\varphi}{\partial {q_{j[J]}}^2} + \sum_{k[J]\neq n_J}\frac{\partial^2\varphi}{\partial q_{k[J]}\partial q_{j[J]}}\right)$$

We could spend more effort to get an orthogonal transformation, but that is not necessary.

We are interested in solutions in which the scalar depends only on the efforts or on the zero frequency components in the relative strategies:

$$\begin{aligned}
&\varphi = \tilde{\varphi}_{j_0} e^{\lambda q_{j_0}} : j_0 \in o[J], j_0 \neq n_J \\
&\sum_K \frac{1}{n_K} \frac{\partial^2 \varphi}{\partial {r_K}^2} + \left(1 - \frac{1}{n_J}\right)^2 \sum_{j[J]\neq n_J} \left(\frac{\partial^2 \varphi}{\partial {q_{j[J]}}^2} + \sum_{k[J]\neq n_J} \frac{\partial^2 \varphi}{\partial q_{k[J]} \partial q_{j[J]}} \right) + \omega(r_1,...,r_N)^2 \varphi = 0 \\
&\sum_K \frac{1}{n_K} \frac{\partial^2 \tilde{\varphi}_{j_0}}{\partial {r_K}^2} + \left(2\lambda^2 \frac{(n_J - 1)^2}{{n_J}^2} + \omega(r_1,...,r_N)^2 \right) \tilde{\varphi}_{j_0} = 0 \\
&\varphi = \tilde{\varphi}_{j_{n_J}} e^{\lambda q_{n_J}} \\
&\sum_K \frac{1}{n_K} \frac{\partial^2 \tilde{\varphi}_{j_{n_J}}}{\partial {r_K}^2} + \left(\lambda^2 \frac{(n_J - 1)^3}{n_J} + \omega(r_1,...,r_N)^2 \right) \tilde{\varphi}_{j_{n_J}} = 0
\end{aligned} \quad . (1.101)$$

As long as the frequency for the relative strategies is chosen to be zero (such as linear behavior), we have the following wave equations:

$$\sum_K \frac{1}{n_K} \frac{\partial^2 \tilde{\varphi}_{j[J]}}{\partial {r_K}^2} + \omega(r_1,...,r_N)^2 \tilde{\varphi}_{j[J]} = 0 . \qquad (1.102)$$

This makes life easy. For each relative strategy, the boundary condition will be

$$\begin{aligned}
\tilde{\varphi}_{j[J]}\Big| &= 1 \Rightarrow \tilde{\varphi}_{j[J]} \equiv \tilde{\varphi}_0 \\
\varphi_{j[J]}\Big| &= q_{j[J]}
\end{aligned} \quad . \qquad (1.103)$$

The boundary condition for the efforts will be the same form of the wave equation and

$$\varphi\Big| = \tilde{\varphi}_{r_J}\Big| = r_J . \qquad (1.104)$$

Thus we solve the same wave equation with different boundary conditions, either the effort direction or unity.

The solution for the coordinates in the original direction will then be:

$$x^{a[J]} = \tilde{\varphi}_{r_J} + \tilde{\varphi}_0 q_{a[J]} \qquad (1.105)$$

For N players we have $N+1$ wave solutions to determine.